

A class of non-supersymmetric orientifolds

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ABSTRACT: We study type IIB orientifolds on T^{2d}/\mathbb{Z}_N with supersymmetry broken by the compactification. We determine tadpole cancellation conditions including antibranes and considering different actions for the parity Ω . Using these conditions we then obtain the spectrum of tachyons and massless states. Various examples with N even correspond to type 0B orientifolds.

KEYWORDS: Superstring Vacua, D-branes.

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1. Introduction

Several ways to build non-supersymmetric open string models have been developed in recent years [1]. One method is to start with orientifolds of the ten-dimensional type 0 strings. The purpose of this note is to pursue this approach further. To this end we will discuss a class of type IIB orientifolds on T^{2d}/\mathbb{Z}_N in which supersymmetry is completely broken by the \mathbb{Z}_N generator θ . The underlying idea is that when no compactification is involved, and θ is a \mathbb{Z}_2 twist corresponding to $(-1)^{F_S}$, where F_S is space-time fermion number, one obtains the ten-dimensional orientifolds of type 0B string that were first studied in [2–4]. Some lower-dimensional cases can be regarded as compactifications of the $D=10$ type 0B orientifolds but the generic situation is just that of a non-supersymmetric type IIB orientifold in $D=(10 - 2d)$ dimensions. Our motivation is to provide a unified description of these constructions. We will consider both the standard world-sheet parity Ω and the modified Ω' defined as

$$\Omega' = \Omega(-1)^{f_R} , \tag{1.1}$$

where f_R is the right-moving world-sheet fermion number. In $D=10$, Ω' leads to a tachyon free model [3] and this can also occur upon compactification for some specific Abelian twists [5–9].

As usual, the orientifold projection introduces Op -planes whose RR charges have to be cancelled by adding appropriate Dp -branes [10, 11]. Since we are breaking supersymmetry, we might as well add anti Dp -branes (\overline{Dp}), just as in toroidal orientifolds in which \mathbb{Z}_N preserves supersymmetry [12]. In general, there is a further sign freedom in the Möbius strip amplitude, or equivalently in the charge and tension of the Op -planes, that allows either an orthogonal or a symplectic projection on the gauge Chan-Paton factors [11, 13]. For instance, in $D=10$, with Ω and no $(-1)^{F_S}$ twist, and including k $\overline{D9}$ -branes, there can be type I-like models with group $SO(32 + k) \times SO(k)$, or $USp(k-32) \times USp(k)$ Sugimoto models [14]. Including the $(-1)^{F_S}$ twist gives instead the $[SO(k_1) \times SO(k_2)]^2$ models of [2–4], with cancelled NSNS tadpoles if $k_1 + k_2 = 32$, or new $[USp(k_1) \times USp(k_2)]^2$ non-supersymmetric orientifolds.

In this work we will explain the general construction of $D \leq 10$ examples. In particular, we will derive tadpole cancellation conditions valid in non-supersymmetric as well as in supersymmetric orientifolds studied in the past [15–20]. For lower dimensional Dp -branes our results carefully take into account their location in the transverse directions. As remarked in [21], only such complete tadpole conditions can guarantee absence of anomalies.

Since the \mathbb{Z}_N action breaks supersymmetry explicitly, the resulting gravity and gauge fields are purely bosonic. However, an interesting property of these models is the appearance of massless charged fermions in the open string sector. The spectrum typically contains tachyons from the closed twisted sectors and also from open string sectors when \overline{Dp} -branes are present. Moreover, there are generically uncanceled NSNS tadpoles although they could be absent in some cases. Nonetheless, we believe that this class of non-supersymmetric orientifolds deserves further investigation. Tachyons and NSNS tadpoles are a common feature of many non-supersymmetric orientifolds but it is conceivable that a consistent theory could emerge after some stabilization process.

This note is organized as follows. In section 2 we review the building blocks of our non-supersymmetric orientifolds. In the next sections we analyze the various orientifolds classified according to the Op -planes present. In each case we first obtain tadpole cancellation conditions and then provide the spectrum of tachyonic and massless states in several examples. The 1-loop amplitudes needed to extract the RR tadpoles are explained in the appendix. Some final remarks are stated in section 6.

2. Generalities

In this section we go over the basic concepts and notation needed to describe our class of models. We consider orientifolds with quotient group $\mathcal{G} = (\mathbb{1} + \Omega_P)\mathbb{Z}_N$, where Ω_P is either the standard Ω or Ω' . More details about the \mathbb{Z}_N action will be presented in section 2.1.

Taking the quotient by \mathcal{G} introduces non-orientable Riemann surfaces in the perturbative expansion. The one-loop vacuum-to-vacuum amplitude on the Klein bottle (\mathcal{K}) has generic tadpole divergences created by orientifold planes charged under RR potentials. The natural way to cancel such tadpoles is to add Dp -branes of opposite RR charges. Open strings with ends on branes have one-loop amplitudes on the cylinder (\mathcal{C}_{pq}) and the Möbius strip (\mathcal{M}_p) whose divergences cancel that of \mathcal{K} [10].

In the appendix we will discuss the 1-loop amplitudes to some extent. We begin with the Klein bottle whose structure determines the orientifold planes and the type of D-branes to be added. Furthermore, the difference between Ω and Ω' manifests only in \mathcal{K} . The cylinder and Möbius strip amplitudes will also be explained.

From the one-loop amplitudes we can deduce the spectrum of tachyons and massless states. General properties of closed and open string states will be described in sections 2.2 and 2.3.

2.1 Non-supersymmetric \mathbb{Z}_N

To describe the action of the \mathbb{Z}_N generator θ on bosonic and fermionic movers X^M, ψ^M in the light cone ($M = 2, \dots, 9$), it is convenient to use complex basis $Y^a = X^{2a+2} + iX^{2a+3}$ and $\Psi^a = \psi^{2a+2} + i\psi^{2a+3}$, $a = 0, \dots, 3$, in which θ is diagonal, i.e. $\theta Y^a = e^{2i\pi v_a} Y^a$ and $\theta \Psi^a = e^{2i\pi v_a} \Psi^a$, where $Nv_a \in \mathbb{Z}$ because $\theta^N = \mathbb{1}$. Space-time is (Minkowski) ^{D} with $D = 10 - 2d$, and the internal space is the orbifold T^{2d}/\mathbb{Z}_N , with coordinates Y^i , $i = 4 - d, \dots, 3$. In the cases of interest, $d \leq 3$. Notice that Lorentz invariance requires $v_a \in \mathbb{Z}$ for $a = 0, \dots, 3 - d$. The T^{2d} lattice is denoted Λ . Recall that windings W take values in Λ , whereas quantized momenta P belong to the dual Λ^* . Under Ω , $W \rightarrow -W$ and $P \rightarrow P$.

Upon compactification, the little group $SO(8)$ breaks to $SO(D-2) \times SO(2d)$ so that string states are classified in terms of $SO(D-2)$ representations. For example, in the right Neveu-Schwarz (NS) sector of closed strings the massless states $\psi_{-\frac{1}{2}}^M |0\rangle$ form an $SO(8)$ vector with weights $\mathbf{8}_v = (\underline{\pm 1}, 0, 0, 0)$, where underline stands for permutations. Under $SO(D-2)$, $\mathbf{8}_v$ branches into a vector and d complex scalars $\Psi_{-\frac{1}{2}}^i |0\rangle$. In the Ramond (R) sector, the vacuum is massless and is given by an $SO(8)$ spinor with weights $\mathbf{8}_s =$

$\pm(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In general, each state has a weight r of $SO(8)$. The GSO projection is $\sum_a r_a = \text{odd}$. Besides, the action of θ is simply $\theta|r\rangle = e^{2i\pi r \cdot v}|r\rangle$, where $v = (v_0, v_1, v_2, v_3)$ is the twist vector.

The v_a 's are constrained by the condition that θ^m must act crystallographically on the lattice Λ of the internal torus. In addition, modular invariance imposes

$$NS_v = \text{even} \quad ; \quad S_v \equiv \sum_a v_a \quad , \quad (2.1)$$

which in turn ensures that θ is of order N acting on the world-sheet fermionic degrees of freedom. Supersymmetry requires the existence of invariant states $|r\rangle$, with r an spinorial weight in $\mathbf{8}_s$. For $v_0 = 0$ this gives the well known condition

$$\pm v_1 \pm v_2 \pm v_3 = 0 \text{ mod } 2. \quad (2.2)$$

Relaxing this condition breaks supersymmetry. The allowed non-supersymmetric twist vectors for $N \leq 6$ were found in [22]. They are:

\mathbb{Z}_2^*	$(0, 0, 0, 1)$	\mathbb{Z}_6^*	$(0, 0, 0, \frac{1}{3})$	(2.3)
\mathbb{Z}_3^*	$(0, 0, 0, \frac{2}{3})$		$(0, 0, \frac{1}{3}, \frac{2}{3})$	
\mathbb{Z}_4	$(0, 0, 0, \frac{1}{2})$		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	
	$(0, 0, \frac{1}{4}, \frac{3}{4})$	\mathbb{Z}_6	$(0, 0, \frac{1}{6}, \frac{1}{2})$	
	$(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		$(0, 0, \frac{1}{6}, \frac{5}{6})$	
\mathbb{Z}_5^*	$(0, 0, \frac{1}{5}, \frac{3}{5})$		$(0, \frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	
			$(0, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$	

For larger N there are many more allowed twists. For the torus lattice for each action we will take products of two-dimensional sub-lattices whenever allowed. Concretely, for order two and order four rotations we take the $SO(4)$ root lattice whereas for order three and order six rotations we take the $SU(3)$ root lattice. The \mathbb{Z}_5 action is realized on the $SU(5)$ root lattice.

In the examples denoted \mathbb{Z}_N^* , $\theta^m Y^i \neq -Y^i$, so that $\Omega\theta^m$ leaves no sub-lattice invariant and the resulting orientifolds will contain only O9-planes. The \mathbb{Z}_2^* gives actually an orientifold of type 0B in $D=10$ since v_3 is integer and in fact $\theta = (-1)^{F_S}$. Notice also that in the \mathbb{Z}_6^* , $3v_a$ is an integer and $\theta^3 = (-1)^{F_S}$.

For the remaining actions in (2.3) there will also be lower dimensional O-planes. Concretely, in all \mathbb{Z}_6 and in the \mathbb{Z}_4 in $D=6$, there are O5₁-planes, with Y^2 and Y^3 transverse, because the element $\theta^{\frac{N}{2}}$ reflects these two coordinates and leaves Y^1 invariant. The \mathbb{Z}_4 in $D=8$ has O7₃-planes, with Y^3 transverse, since $\theta Y^3 = -Y^3$. Finally, the \mathbb{Z}_4 in $D=4$ will include O3-planes because θ reflects all internal coordinates.

2.2 Closed string states

Modular invariance requires the existence of sectors twisted by θ^n , $n = 0, \dots, N - 1$. In each sector the states are tensor products $|R\rangle \times |L\rangle$ of right and left modes. In turn,

$|R\rangle = |N_R, r_{nR}\rangle$, where N_R is an oscillator number and $r_{nR} = r_R + nv$ with r_R an $SO(8)$ weight (similar for $|L\rangle$). To determine the states in the orientifold spectrum, we start with invariant combinations under \mathbb{Z}_N . For example, in the untwisted sector it suffices to have $(r_R - r_L) \cdot v = \text{integer}$ for states without oscillators acting on them. In the twisted sectors we must take into account the structure of fixed points that depends on the torus lattice, see the appendix of [22] for more details.

We then implement the orientifold projection, i.e. invariance under Ω_P . In the untwisted sector both Ω and Ω' just exchange left and right modes. Then, the invariant combinations must be symmetric in the NSNS and [NSR + RNS] sectors, but antisymmetric in the RR sector since fermionic modes are exchanged. In the $\theta^{\frac{N}{2}}$ sector, when N is even, the invariant combinations are the same for $\Omega_P = \Omega$. On the other hand, as explained in section A, for $\Omega_P = \Omega'$, when $\frac{N}{2}v$ is such that $\theta^{\frac{N}{2}}$ is equivalent to $(-1)^{F_S}$, there is an extra minus sign in the Klein-bottle amplitude so that in the $\theta^{\frac{N}{2}}$ sector we need take the opposite type of combinations as in the untwisted sector. For the remaining twisted sectors we have to take into account that $\theta^n \rightarrow \theta^{N-n}$ under Ω_P .

2.3 Open string states

Including labels ab for the ends on Dp and Dq -branes, states are of the form

$$|\phi, ab\rangle(\lambda_{pq}^\phi)_{ab}, \tag{2.4}$$

where λ_{pq}^ϕ is the Chan-Paton matrix and ϕ represents the world-sheet modes. The action of θ^m and $\Omega_P \theta^m$ on the Chan-Paton matrices is realized by the unitary matrices $\gamma_{m,p}$ and $\gamma_{\Omega m,p}$ such that

$$\theta^m : \lambda_{pq}^\phi \rightarrow \gamma_{m,p} \lambda_{pq}^\phi \gamma_{m,q}^{-1} \quad ; \quad \Omega_P \theta^m : \lambda_{pq}^\phi \rightarrow \gamma_{\Omega m,q} \lambda_{pq}^{\phi T} \gamma_{\Omega m,p}^{-1}, \tag{2.5}$$

since Ω_P exchanges the ends.

The γ matrices form a representation of \mathcal{G} up to a phase [15]. Consistent with group multiplication we can define

$$\gamma_{m,p} = \gamma_{1,p}^m \quad ; \quad \gamma_{\Omega m,p} = \gamma_{m,p} \gamma_{\Omega,p}. \tag{2.6}$$

It can be shown that $\gamma_{0,p} = \mathbb{1}$ [15]. Besides, $\theta^N = \mathbb{1}$, gives

$$\gamma_{N,p} = (\gamma_{1,p})^N = \delta_p \mathbb{1}, \tag{2.7}$$

where $\delta_p = \pm 1$, keeping the sign freedom.

When only D9 and D5-branes are present, it happens that $(\Omega_P \theta^m)^2 = \theta^{2m}$ acting on pp strings. It then follows that

$$\gamma_{\Omega m,p} = \epsilon_{m,p} \gamma_{2m,p} \gamma_{\Omega m,p}^T, \tag{2.8}$$

where $\epsilon_{m,p} = \pm 1$. Now, we can always absorb a phase in $\gamma_{1,p}$ to attain

$$\epsilon_{m,p} = \epsilon_{0,p} \equiv \epsilon_p. \tag{2.9}$$

It is then useful to recast (2.8) as

$$\gamma_{m,p}^* = \epsilon_p \gamma_{\Omega,p}^* \gamma_{m,p} \gamma_{\Omega,p} . \tag{2.10}$$

The γ matrices are further constrained by tadpole cancellation conditions. For antibranes, $\delta_{\bar{p}} = \delta_p$ and $\epsilon_{\bar{p}} = \epsilon_p$. In section 5 we will explain how these results are modified when D3 or D7-branes are included.

To understand the meaning of ϵ_p , first use (2.8) to derive the relations

$$\gamma_{\Omega,p}^T = \epsilon_p \gamma_{\Omega,p} \quad ; \quad \text{Tr} \left(\gamma_{\Omega,p}^{-1} \gamma_{\Omega,p}^T \right) = \epsilon_p N_p , \tag{2.11}$$

where N_p is the total number of D p -branes. For instance, when $\epsilon_p = 1$ we can take $\gamma_{\Omega,p} = \mathbb{1}$, leading to orthogonal projection $\lambda^T = -\lambda$ on the Chan-Paton matrix of pp gauge vectors (see below). For $\epsilon_p = -1$ there is instead a symplectic projection. The second relation in (2.11) means that ϵ_p correlates with the sign of the tension and the RR charge of Op -planes. In particular, $\epsilon_p = 1$ corresponds to Op -planes of negative tension and RR charge. For $\epsilon_p = -1$ both signs are reversed. In the notation of [11] these are $Op+$ and $Op-$ respectively.

To determine the world sheet modes ϕ , and then find the states invariant under \mathcal{G} , it is necessary to specify the type of branes at the ends. For states coming from open strings ending at the same type of branes we need to look at one-loop cylinder and Möbius strip amplitudes \mathcal{C}_{pp} and \mathcal{M}_p , as well as $\mathcal{C}_{\bar{p}\bar{p}}$ and $\mathcal{M}_{\bar{p}}$ if there are antibranes. For open strings ending at different branes it suffices to study \mathcal{C}_{pq} , $\mathcal{C}_{\bar{p}\bar{q}}$ and $\mathcal{C}_{p\bar{q}}$. Since Ω_P exchanges the ends, these open strings cannot enter in Möbius strip traces. We give explicit results for configurations with D9 + $\bar{D}9$ that are generically present, and also with D5 + $\bar{D}5$ that appear in several \mathbb{Z}_{even} examples. There are cases with D7 + $\bar{D}7$ or D3 + $\bar{D}3$ that will be explained when discussing the concrete examples.

2.3.1 99, $\bar{9}\bar{9}$ and $9\bar{9} + \bar{9}9$ states

For 99 strings, ϕ is given by the right modes of the closed string. To each state we assign an SO(8) weight r verifying the GSO projection $\sum_a r_a = \text{odd}$ that eliminates the tachyon and leaves massless states $r = \mathbf{8}_v$ in the NS sector and $r = \mathbf{8}_s$ in the R sector. The Chan-Paton matrix λ_{99}^r of the massless invariant states must satisfy

$$\lambda_{99}^r = e^{2i\pi r \cdot v} \gamma_{1,9}^r \lambda_{99}^r \gamma_{1,9}^{-1} \quad , \quad \lambda_{99}^r = -\gamma_{\Omega,9} \lambda_{99}^r \gamma_{\Omega,9}^{-1} . \tag{2.12}$$

For $\bar{9}\bar{9}$ strings, the cylinder partition function is identical to that of 99 strings, thus the massless weights are the same. The invariance conditions are analogous to (2.12) except for an extra minus sign in the Ω_P projection for the R states. This is due to the sign change in the R sector of the Möbius strip amplitude $\mathcal{M}_{\bar{9}}$, which in turn corresponds to the opposite RR charge of $\bar{D}9$ -branes.

The cylinder amplitude $\mathcal{C}_{9\bar{9}}$ is obtained from \mathcal{C}_{99} by reversing the relative sign between the two NS (and the two R) contributions. We can still assign SO(8) weights to the various states but the GSO projection changes to $\sum_a r_a = \text{even}$. The NS sector then contains a tachyon with $r = 0$ and no massless states. In the R sector there is a massless spinor of

opposite chirality, with weights $r = \mathbf{8}_c = \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\underline{-\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Invariance under θ implies:

$$\lambda_{9\bar{9}}^r = e^{2i\pi r \cdot v} \gamma_{1,9} \lambda_{9\bar{9}}^r \gamma_{1,\bar{9}}^{-1}. \quad (2.13)$$

Under Ω_P , $9\bar{9} \rightarrow \bar{9}9$, so that we must just retain half of the states in the spectrum without further restrictions.

To find the states arising under compactification we just decompose the $\text{SO}(8)$ weights under $\text{SO}(D-2) \times \text{SO}(2d)$. From $\mathbf{8}_v$ we clearly obtain a vector, with $r \cdot v = 0$, plus scalars with $r \cdot v = \pm v_i$. In the 99 and $\bar{9}\bar{9}$ sectors the fermions come from $\mathbf{8}_s$ which in the different dimensions decomposes as

$$\begin{aligned} D = 8 : \quad \mathbf{8}_s &= (\mathbf{4}, -\frac{1}{2}) + (\bar{\mathbf{4}}, \frac{1}{2}), \\ D = 6 : \quad \mathbf{8}_s &= (\mathbf{2}_L, \mathbf{2}_R) + (\mathbf{2}_R, \mathbf{2}_L), \\ D = 4 : \quad \mathbf{8}_s &= (\frac{1}{2}, \bar{\mathbf{4}}) + (-\frac{1}{2}, \mathbf{4}). \end{aligned} \quad (2.14)$$

We use conventions such that for $\text{SO}(6)$, $\mathbf{4} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\underline{-\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2})$. Notice that the $\text{SO}(2)$ charge in $D=8$ is the component r_3 of r , whereas in $D=4$ it is r_0 which corresponds to helicity. For the $\text{SO}(4)$ representations in $D=6$ our convention is $\mathbf{2}_L = \pm(\frac{1}{2}, -\frac{1}{2})$ and $\mathbf{2}_R = \pm(\frac{1}{2}, \frac{1}{2})$. In $D=8, 4$ we will distinguish fermions from antifermions by the condition $\prod_{i=(D-2)/2}^3 r_i < 0$.

When the rotation breaks supersymmetry, none of the 99 or $\bar{9}\bar{9}$ fermions has weights with $r \cdot v = 0$. Thus, there are no gauginos to pair up with gauge vectors. Instead, there are charged fermions whose $r \cdot v$ follows from the branching rules in (2.14). For example, in $D=8$ the fermions belong to the $\mathbf{4}$ of $\text{SO}(6)$ and have $r \cdot v = -v_3/2$ (antiparticles have the opposite). In $D=4$ the fermionic particles have helicity $1/2$ and come in four types according to $r \cdot v = -S_v/2$ or $r \cdot v = -v_i + S_v/2$. In $D=6$ there are fermions in $\mathbf{2}_L$ with $r \cdot v = S_v/2$ and in $\mathbf{2}_R$ with $r \cdot v = -v_3 + S_v/2$, plus the complex conjugates. In the $9\bar{9}$ sector the $D=10$ massless spinor is $\mathbf{8}_c$. Hence, the fermions arising from this sector will have opposite chiralities.

2.3.2 $55, \bar{5}\bar{5}$ and $5\bar{5} + \bar{5}5$ states

For concreteness we focus on $D5_1$ -branes. Then the coordinate Y^1 has NN whereas $Y^{2,3}$ have DD boundary conditions. Since there are no mixed boundary conditions, the massless 55 states can be labelled by $\text{SO}(8)$ weights, $\mathbf{8}_v$ in NS and $\mathbf{8}_s$ in R, that must be appropriately decomposed upon compactification. It is necessary to distinguish the states associated to directions with DD or NN boundary conditions. For $D5_1$ -branes the corresponding r 's are

$$\begin{aligned} r_v^{\text{DD}} &= (0, 0, \underline{\mp 1}, 0) \quad ; \quad r_s^{\text{DD}} = \pm(\frac{1}{2}, \frac{1}{2}, \underline{-\frac{1}{2}}, \frac{1}{2}) \\ r_v^{\text{NN}} &= (\underline{\mp 1}, 0, 0, 0) \quad ; \quad r_s^{\text{NN}} = \pm(\underline{-\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}). \end{aligned} \quad (2.15)$$

This is valid when $\frac{N}{2}S_v = \text{even}$, otherwise r_s^{DD} and r_s^{NN} are exchanged. The point is that in supersymmetric \mathbb{Z}_N some or all of the r_s^{NN} correspond to gauge spinors that must satisfy $r_s^{\text{NN}} \cdot v \in \mathbb{Z}$ in order to pair with gauge vectors having $r_v^{\text{NN}} \cdot v = 0$.

The full orientifold projection requires that the Chan-Paton factors λ_{55}^r satisfy

$$\lambda_{55}^r = e^{2i\pi r \cdot v} \gamma_{1,5,J} \lambda_{55}^r \gamma_{1,5,J}^{-1} \quad , \quad \lambda_{55}^r = \pm \gamma_{\Omega,5} \lambda_{55}^{rT} \gamma_{\Omega,5}^{-1} . \quad (2.16)$$

The index J in $\gamma_{1,5,J}$ refers to the fixed point of θ where the $D5_1$ -branes are located. When the branes sit at a point in the bulk there is no orbifold projection since θ only exchanges images. In the Ω projection, the plus sign applies to states associated with DD boundary conditions. Notice that for fermions, DD or NN is correlated with the chirality of the spinor in the brane world-volume. If the branes sit at a fixed point of θ^m , but not of θ , in (2.16) we need replace $v \rightarrow mv$ and $\gamma_{1,5,J} \rightarrow \gamma_{m,5,J}$.

For $\bar{5}\bar{5}$ states the massless weights are the same as in (2.15). The projection is similar to (2.16) except for an extra minus sign, due to the opposite RR charge, in the Ω projection of R states.

In the $5\bar{5}$ sector the GSO projection allows instead a tachyon with $r = 0$ and a massless spinor $\mathbf{8}_c$ of opposite chirality. We only need to impose the θ projection

$$\lambda_{5\bar{5}}^r = e^{2i\pi r \cdot v} \gamma_{1,5,J} \lambda_{5\bar{5}}^r \gamma_{1,5,J}^{-1} . \quad (2.17)$$

The Ω projection just exchanges states in $5\bar{5}$ with those in $\bar{5}5$.

2.3.3 $95 + 59, \bar{9}\bar{5} + \bar{5}\bar{9}, 9\bar{5} + \bar{5}9$ and $\bar{9}5 + 5\bar{9}$ states

It is enough to discuss 95 states in some detail. The new feature is the appearance of mixed boundary conditions. For $D5_1$ -branes, $Y^{0,1}$ are NN but $Y^{2,3}$ are ND. As explained in appendix A, the partition function changes in such a way that massless states in the NS and R sector have weights

$$r_{\text{NS}} = (0, 0, w_2, w_3) \quad ; \quad r_{\text{R}} = (w_0, w_1, 0, 0) . \quad (2.18)$$

The w_a take values $\pm \frac{1}{2}$ and are constrained by the GSO projection that depends on whether $\frac{N}{2}S_v$ is even or odd. Concretely, for $\frac{N}{2}S_v$ even, $\sum_a w_a = \text{odd}$, but for $\frac{N}{2}S_v$ odd, $\sum_a w_a = \text{even}$. The θ projection is simply

$$\lambda_{95}^r = e^{2i\pi r \cdot v} \gamma_{1,9} \lambda_{95}^r \gamma_{1,5,J}^{-1} . \quad (2.19)$$

The Ω projection only effects $95 \leftrightarrow 59$.

For $\bar{9}\bar{5}$ states the massless weights are the same as for 95. For $\bar{9}5$ and $9\bar{5}$ the massless weights have the same form as in (2.18) but the GSO projection is the opposite compared to 95. In all cases the θ projection is analogous to (2.19).

3. Models with O9-planes

This is the case of the \mathbb{Z}_N^* actions whose elements do not invert any coordinate. To begin we use the results of the appendix to deduce the tadpole cancellation conditions. We then determine the γ matrices that form a representation of \mathcal{G} and cancel RR tadpoles. As explained before, the group structure allows a freedom in $\epsilon_{0,p} = \epsilon_p$ that enters in (2.10).

We will consider both values for ϵ_p . In several examples we will present the resulting spectrum of massless and tachyonic states for both Ω and Ω' projections.

An useful check on the spectrum is the cancellation of the irreducible gauge anomaly proportional to $\text{tr} F^{\frac{D+2}{2}}$ for each group factor. In $D = 10, 6$, the irreducible gravitational anomaly must vanish as well. In $D=6$, cancellation of the $\text{tr} R^4$ anomaly requires

$$N_L - N_R = 28(n_- - n_+) , \tag{3.1}$$

where n_{\pm} is the number of tensors $\mathbf{3}^{\pm}$ whereas N_L and N_R are the numbers of $2(\mathbf{2}_L)$ and $2(\mathbf{2}_R)$ respectively.

3.1 Tadpole cancellation with O9-planes

For the \mathbb{Z}_N^* actions there are only untwisted tadpoles proportional to a 10-dimensional volume V_{10} , and twisted tadpoles proportional to V_D . The reason is that the invariant momentum sub-lattice is either trivial and its volume is $V_P = 1$ by definition, or it is the full Λ^* and $V_D V_P$ is V_{10} . We are employing the notation introduced in the appendix.

Let us first consider the divergences from the Klein bottle amplitude. When $\Omega_P = \Omega$, the RR tadpole (A.9) simplifies to

$$T_{\mathcal{K}}^{\text{RR}}(n, m) = e^{-i\pi n S_v} 2^{D+I_P} V_D V_P \prod_{2mv_j \notin \mathbb{Z}} 2 |\sin 2\pi mv_j| . \tag{3.2}$$

For the NSNS tadpoles there is an analogous formula but with the phase $e^{-i\pi n S_v}$ absent. When N is even, the RR tadpoles vanish since

$$\sum_{m=0}^{N-1} [T_{\mathcal{K}}^{\text{RR}}(0, m) + T_{\mathcal{K}}^{\text{RR}}(\frac{N}{2}, m)] = 0 . \tag{3.3}$$

However, there are left-over NSNS tadpoles, either proportional to V_{10} or to V_D , that can be cancelled by introducing D9-branes. The RR tadpoles created by the D9-branes can in turn be cancelled by adding $\overline{\text{D9}}$ -branes. For N odd adding D9-branes is mandatory since there are uncancelled RR tadpoles.

When $\Omega_P = \Omega'$, there is an extra minus sign in $T_{\mathcal{K}}^{\text{RR}}(\frac{N}{2}, m)$. Thus, the divergences from the untwisted and the $\frac{N}{2}$ -twisted sector add to produce a non-zero RR tadpole. On the other hand, the NSNS tadpoles from both sectors cancel each other.

From the \mathcal{C}_{99} cylinder the RR tadpole (A.18) reduces to:

$$T_{99}^{\text{RR}}(m) = (\text{Tr} \gamma_{m,9})^2 V_D V_P \prod_{mv_j \notin \mathbb{Z}} 2 |\sin \pi mv_j| . \tag{3.4}$$

In the presence of $\overline{\text{D9}}$ -branes, we need to substitute $\text{Tr} \gamma_{m,9}$ by $\text{Tr} \gamma_{m,9} - \text{Tr} \gamma_{m,\bar{9}}$.

In \mathcal{M}_9 we find:

$$T_9^{\text{RR}}(m) = -2^{\frac{1}{2}(D+I_P+2)} (\epsilon_9 \text{Tr} \gamma_{2m,9}) V_D V_P \prod_{a=0}^3 c(mv_a) \prod_{mv_j \notin \mathbb{Z}} 2 |\sin \pi mv_j| \tag{3.5}$$

where we have used (2.8). For $\mathcal{M}_{\bar{9}}$ there is an extra minus sign due to the opposite RR charge.

Clearly, the tadpoles (3.2), (3.4) and (3.5) have a definite volume dependence. In fact, they are proportional either to V_{10} or to V_D . Furthermore, to derive the cancellation conditions the tadpoles from the various amplitudes are combined according to the twisted sector in the transverse (tree-level) channel. For N odd and $\forall m$ the RR tadpoles cancel provided that

$$\mathrm{Tr} \gamma_{2m,9} - \mathrm{Tr} \gamma_{2m,\bar{9}} = 32 \epsilon \prod_{j=1}^d \cos m\pi v_j, \quad (3.6)$$

where $\epsilon \equiv \epsilon_9 = \epsilon_{\bar{9}}$. Additionally, we can choose $\gamma_{N,p} = \mathbb{1}$ ($p = 9, \bar{9}$) without loss of generality. In case of orthogonal projection, $\epsilon = 1$, we can avoid the $\overline{\mathrm{D9}}$ -branes. For the symplectic projection, $\epsilon = -1$, the O9-planes have positive RR charge and $\overline{\mathrm{D9}}$ -branes are necessary. Concerning NSNS tadpoles, for the \mathbb{Z}_3 and \mathbb{Z}_5 in (2.3) we have found that they vanish when the RR tadpoles do.

Let us now consider N even. For $\Omega_P = \Omega$, cancellation of RR tadpoles in \mathcal{K} shows the existence of two types of orientifold planes with opposite charges. This interpretation is possible because in \mathbb{Z}_N^* orientifolds there are two sets of RR potentials, one set from the untwisted sector and the other from the $\theta^{\frac{N}{2}}$ sector. By consistency, the RR Möbius tadpoles must also cancel. This requires $\gamma_{N,p} = \mathbb{1}$, for $p = 9, \bar{9}$. Additionally, to cancel the cylinder tadpoles,

$$\mathrm{Tr} \gamma_{m,9} - \mathrm{Tr} \gamma_{m,\bar{9}} = 0 \quad , \quad \forall m. \quad (3.7)$$

This is the RR tadpole cancellation condition in a type 0B orientifold on $T^{2d}/\tilde{\mathbb{Z}}_{\frac{N}{2}}$, with $\frac{N}{2}$ odd and $\tilde{v} = (v_0, v_1, v_2, v_3 - 1)$. As for NSNS tadpoles, adding the pieces from the different amplitudes we obtain the cancellation condition

$$\mathrm{Tr} \gamma_{2m,9} + \mathrm{Tr} \gamma_{2m,\bar{9}} = 64 \epsilon \prod_{j=1}^d \cos m\pi \tilde{v}_j. \quad (3.8)$$

Notice that for $\epsilon = -1$ the NSNS tadpoles cannot be cancelled at all. This is as expected since in this case O9-planes have positive tension as D9-branes and $\overline{\mathrm{D9}}$ -branes do. However, for $\epsilon = 1$, both RR and NSNS tadpoles can vanish as we will exemplify later.

We finally come to N even and $\Omega_P = \Omega'$. Now there is a RR tadpole from the Klein bottle and necessarily from the Möbius amplitude. Then, it must be $\gamma_{N,p} = -\mathbb{1}$, for $p = 9, \bar{9}$. Collecting all pieces we obtain the RR tadpole cancellation conditions

$$\begin{aligned} \mathrm{Tr} \gamma_{2m+1,9} - \mathrm{Tr} \gamma_{2m+1,\bar{9}} &= 0 \\ \mathrm{Tr} \gamma_{2m,9} - \mathrm{Tr} \gamma_{2m,\bar{9}} &= 64 \epsilon \prod_{j=1}^d \cos m\pi v_j, \end{aligned} \quad (3.9)$$

both for $m = 0, \dots, \frac{N}{2} - 1$. There are always NSNS tadpoles left uncanceled.

3.2 Solutions with Ω projection

As we have seen in 3.1, for N even necessarily $\gamma_{1,p}^N = \mathbb{1}$ ($p = 9, \bar{9}$) and for N odd we can make the same choice without altering the results. Then, in general ($\mu = e^{2i\pi/N}$)

$$\gamma_{1,9} = \text{diag}(\mathbb{1}_{n_0}, \mu \mathbb{1}_{n_1}, \mu^2 \mathbb{1}_{n_2}, \dots, \mu^{N-1} \mathbb{1}_{n_{N-1}}) \quad ; \quad n_{N-j} = n_j, \quad (3.10)$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix. For $\gamma_{1,\bar{9}}$, replace n_j by \bar{n}_j . Imposing RR tadpole cancellation relates the n_j to the \bar{n}_j . For instance, for $N = 2L + 1$, (3.6) becomes

$$(n_0 - \bar{n}_0) + 2 \sum_{k=1}^L (n_k - \bar{n}_k) \cos \frac{4\pi mk}{N} = 32 \epsilon \prod_{j=1}^d \cos(m\pi v_j) \quad ; \quad m = 0, \dots, L. \quad (3.11)$$

For $N = 2L$, (3.7) simply gives $n_j = \bar{n}_j$. To cancel NSNS tadpoles, according to (3.8), there are further conditions

$$n_0 + n_L + 2 \sum_{k=1}^{L-1} n_k \cos \frac{4\pi mk}{N} = 32 \epsilon \prod_{j=1}^d \cos(m\pi \tilde{v}_j) \quad ; \quad m = 0, \dots, L-1. \quad (3.12)$$

Clearly, this needs $\epsilon = 1$.

Concerning the realization of Ω , in agreement with (2.10), for $\epsilon_9 \equiv \epsilon = 1$ we can take

$$\gamma_{\Omega,9}^+ = \begin{pmatrix} \mathbb{1}_{n_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{1}_{n_1} \\ 0 & 0 & 0 & \cdots & \mathbb{1}_{n_2} & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \mathbb{1}_{n_2} & \cdots & 0 & 0 \\ 0 & \mathbb{1}_{n_1} & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (3.13)$$

For $\epsilon = -1$ we need to distinguish between N odd or even. For $N = 2L + 1$,

$$\gamma_{\Omega,9}^- = \begin{pmatrix} iJ_{n_0} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & i\mathbb{1}_{n_1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & i\mathbb{1}_{n_L} & \cdots & 0 \\ 0 & 0 & \cdots & -i\mathbb{1}_{n_L} & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & -i\mathbb{1}_{n_1} & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \quad ; \quad J_{n_0} = \begin{pmatrix} 0 & \mathbb{1}_{n_0/2} \\ -\mathbb{1}_{n_0/2} & 0 \end{pmatrix}. \quad (3.14)$$

Clearly n_0 must be even. Instead, for $N = 2L$,

$$\gamma_{\Omega,9}^- = \begin{pmatrix} iJ_{n_0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & i\mathbb{1}_{n_1} \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & i\mathbb{1}_{n_{L-1}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 & iJ_{n_L} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -i\mathbb{1}_{n_{L-1}} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & -i\mathbb{1}_{n_1} & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (3.15)$$

with both n_0 and n_L even.

Consider now the the open string spectrum. The task is to determine the Chan-Paton factors after substituting the above γ matrices in, say (2.12) for 99 states. The simplest case is that of gauge vectors that have $r \cdot v = 0$. From the form of the resulting λ 's we can read the gauge groups. For $\epsilon = 1$ we find

$$\begin{aligned} N = 2L + 1 & : G_9 = \text{SO}(n_0) \times \text{U}(n_1) \times \cdots \times \text{U}(n_L) , \\ N = 2L & : G_9 = \text{SO}(n_0) \times \text{U}(n_1) \times \cdots \times \text{U}(n_{L-1}) \times \text{SO}(n_L) . \end{aligned} \quad (3.16)$$

The $\bar{9}9$ group factors in $G_{\bar{9}}$ are the same with $n_k \rightarrow \bar{n}_k$. For $\epsilon = -1$, we just need replace $\text{SO}(n) \rightarrow \text{USp}(n)$ and take n even.

The massless charged scalars and fermions transform under θ as explained in section 2.3.1. Given the possible values of $r \cdot v$ we find the Chan-Paton matrices according to (2.12) and then read off the corresponding representations. Specific examples will be presented in section 3.4. In the $\bar{9}9$ sector besides massless charged fermions there are charged tachyons that have $r = 0$. Under $G_9 \times G_{\bar{9}}$ they turn to transform as

$$(\square_0, \square_{\bar{0}}) + \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} [(\square_k, \bar{\square}_k) + \text{c.c.}] + \left(\square_{\frac{N}{2}}, \square_{\frac{\bar{N}}{2}} \right) , \quad (3.17)$$

where the last term appears only if N is even. The subscripts attached to the Young tableaux refer to the k -th group factor in G_9 or in $G_{\bar{9}}$. It is understood that the states are singlets under absent factors.

3.3 Solutions with Ω' projection

As explained in 3.1 we only need consider N even ($N = 2L$) and $\gamma_{1,p}^N = -1$. Then, in general ($\nu = e^{i\pi/N}$)

$$\gamma_{1,9} = \text{diag}(\nu \mathbb{1}_{n_1}, \nu^3 \mathbb{1}_{n_2}, \dots, \nu^{2N-1} \mathbb{1}_{n_N}) \quad ; \quad n_{N-j+1} = n_j . \quad (3.18)$$

The $\gamma_{1,\bar{9}}$ is analogous with $n_j \leftrightarrow \bar{n}_j$. There are relations between the dimensions n_j and \bar{n}_j following from the cancellation conditions (3.9). For instance, for the \mathbb{Z}_2^* , $(n_1 - \bar{n}_1) = 32\epsilon$, whereas for the \mathbb{Z}_6^* in (2.3),

$$(n_1 - \bar{n}_1) = (n_3 - \bar{n}_3) = \frac{32}{3}\epsilon(1 + \prod_j \cos \pi v_j) \quad ; \quad (n_2 - \bar{n}_2) = \frac{32}{3}\epsilon(1 - 2\prod_j \cos \pi v_j) . \quad (3.19)$$

Notice that we can take $\epsilon = -1$ only if $\bar{\text{D}}9$ -branes are present.

The realization of Ω' is given by

$$\gamma_{\Omega,9}^+ = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbb{1}_{n_1} \\ 0 & 0 & \cdots & \mathbb{1}_{n_2} & 0 \\ \vdots & & & & \vdots \\ 0 & \mathbb{1}_{n_2} & \cdots & 0 & 0 \\ \mathbb{1}_{n_1} & 0 & \cdots & 0 & 0 \end{pmatrix} \quad ; \quad \gamma_{\Omega,9}^- = \begin{pmatrix} 0 & 0 & \cdots & 0 & i\mathbb{1}_{n_1} \\ 0 & 0 & \cdots & i\mathbb{1}_{n_2} & 0 \\ \vdots & & & & \vdots \\ 0 & -i\mathbb{1}_{n_2} & \cdots & 0 & 0 \\ -i\mathbb{1}_{n_1} & 0 & \cdots & 0 & 0 \end{pmatrix} , \quad (3.20)$$

for $\epsilon = 1$ and $\epsilon = -1$ respectively.

Given the realization of the full orientifold action we can proceed to find the Chan-Paton factors for the various massless and tachyonic open states. The gauge group for both $\epsilon = \pm 1$ turns out to be

$$G_9 = U(n_1) \times \cdots \times U(n_L) , \tag{3.21}$$

and similarly for $G_{\bar{9}}$. The charged tachyons in the $9\bar{9}$ sector transform under $G_9 \times G_{\bar{9}}$ as

$$\sum_{k=1}^L [(\square_k, \bar{\square}_{\bar{k}}) + \text{c.c.}] . \tag{3.22}$$

The charged massless fermions depend on the specific \mathbb{Z}_N . Examples will be presented in section 3.4. Actually it suffices to consider $\epsilon = 1$, otherwise we just exchange branes and antibranes.

3.4 Examples

In this section we present a more detailed account of closed and open states in some \mathbb{Z}_N^* selected from those in (2.3). Since the generic gauge groups are already given in eqs. (3.16) and (3.21), we will only supply the relations between n_k and \bar{n}_k due to RR tadpole cancellation. To simplify presentation we only display the massless charged fermions. We also provide the closed massless and tachyonic states. In the untwisted sector, denoted θ^0 , we only list the states besides the dilaton plus metric, and the antisymmetric tensor, that are always present among respectively NSNS and RR massless states. In the twisted θ^n sectors we list all states. For N even we will consider the Ω and Ω' projections. The closed spectrum in both cases differ only in the $\theta^{\frac{N}{2}}$ sector.

It is straightforward to work out many more examples. In table 1 we display the results for the remaining \mathbb{Z}_6^* in (2.3), but only with Ω' projection because then D9-branes must be necessarily included and antibranes can be completely removed.

3.4.1 $D=10, \mathbb{Z}_2^*, v = (0, 0, 0, 1)$

With Ω projection this is the type 0B orientifold discussed at length in [2–4]. The RR tadpoles cancel if $n_0 = \bar{n}_0$, and $n_1 = \bar{n}_1$. Clearly, we are free to discard branes and antibranes altogether. However, we will keep them because NSNS tadpoles cancel if $n_0 + n_1 = 32$. The closed spectrum in the twisted sector is:

$$\theta \quad : \quad \mathbf{1}_{-1} \text{ (NSNS)} + \mathbf{28} \text{ (RR)} , \tag{3.23}$$

where states are labelled by their $SO(8)$ representations and $\mathbf{1}_{-1}$ denotes a tachyon of mass -1 ($\alpha'=1$). The charged massless fermions are:

$$\mathbf{8}_s (\square_0, \square_1) + \mathbf{8}_c (\square_0, \square_{\bar{1}}) + k \leftrightarrow \bar{k} . \tag{3.24}$$

For $\epsilon = -1$ there are USp rather than SO groups but the fermions transform in the same way.

With Ω' projection we have instead the type 0B' orientifold first discussed in [3]. The closed spectrum in the twisted sector is now:

$$\theta \quad : \quad [\mathbf{1} + \mathbf{35}^+] (\text{RR}) . \quad (3.25)$$

Comparing with (3.23) we see that the tachyon has disappeared and instead of an anti-symmetric tensor there is a 4-form with self-dual field strength, denoted $\mathbf{35}^+$.

The RR tadpole cancellation condition with the Ω' projection is $n_1 = 32 + \bar{n}_1$. The charged massless fermions are:

$$\mathbf{8}_s \left[\begin{array}{|c|} \hline \square \\ \hline \square_1 \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square_{\bar{1}} \end{array} + \text{c.c.} \right] + \mathbf{8}_c \left[(\square_1, \square_{\bar{1}}) + \text{c.c.} \right] . \quad (3.26)$$

The irreducible $\text{tr } R^6$ anomaly of the 4-form is precisely cancelled by these fermions [3, 23].

3.4.2 $D=8, \mathbb{Z}_3^*, v = (0, 0, 0, \frac{2}{3})$

The closed sector states, identified by $\text{SO}(6)$ representations, are:

$$\begin{aligned} \theta^0 & : \quad \mathbf{1} (\text{NSNS}) + \mathbf{1} (\text{RR}) + \mathbf{4} (\text{NSR}) , \\ \theta + \theta^2 & : \quad 3 \left\{ \left[\mathbf{1}_{-\frac{2}{3}} + \mathbf{1} \right] (\text{NSNS}) + [\mathbf{1} + \mathbf{15}] (\text{RR}) + \mathbf{4} (\text{NSR}) \right\} . \end{aligned} \quad (3.27)$$

With orthogonal projection ($\epsilon = 1$), the RR (and NSNS) tadpoles cancel if $n_0 = \bar{n}_0$, and $n_1 = 16 + \bar{n}_1$. Notice that antibranes can be avoided but we will give the generic open spectrum. The charged massless fermions are:

$$4 \left[\begin{array}{|c|} \hline \square \\ \hline \square_1 \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square_{\bar{1}} \end{array} \right] + \bar{\mathbf{4}} (\square_1, \square_{\bar{1}}) + \left\{ \mathbf{4} (\square_0, \bar{\square}_1) + \bar{\mathbf{4}} (\square_0, \bar{\square}_{\bar{1}}) + k \leftrightarrow \bar{k} \right\} . \quad (3.28)$$

It is easy to check that irreducible $\text{tr } F^5$ gauge anomalies are absent by virtue of RR tadpole cancellation.

With symplectic projection, $G_9 = \text{USp}(n_0) \times \text{U}(n_1)$ and $G_{\bar{9}} = \text{USp}(\bar{n}_0) \times \text{U}(\bar{n}_2)$, with even $\bar{n}_0 = n_0$ and $\bar{n}_1 = 16 + n_1$. Massless fermions are basically those of the orthogonal case upon interchanging representations of G_9 and $G_{\bar{9}}$. The total anomaly factorizes appropriately.

3.4.3 $D=6, \mathbb{Z}_5^*, v = (0, 0, \frac{1}{5}, \frac{3}{5})$

The closed sector states are:

$$\begin{aligned} \theta^0 & : \quad 2(\mathbf{1}) (\text{NSNS}) + 2(\mathbf{1}) (\text{RR}) + 2[\mathbf{2}_L + \mathbf{2}_R] (\text{NSR}) , \\ \theta + \theta^4 & : \quad 5 \left\{ \left[\mathbf{1}_{-\frac{2}{5}} + \mathbf{1} \right] (\text{NSNS}) + [\mathbf{1} + \mathbf{3}^+] (\text{RR}) + 2(\mathbf{2}_L) (\text{NSR}) \right\} , \\ \theta^2 + \theta^3 & : \quad 5 \left\{ \left[\mathbf{1}_{-\frac{2}{5}} + \mathbf{1} \right] (\text{NSNS}) + [\mathbf{1} + \mathbf{3}^-] (\text{RR}) + 2(\mathbf{2}_R) (\text{NSR}) \right\} , \end{aligned} \quad (3.29)$$

Here we label states by their $\text{SO}(4)$ representations and $\mathbf{3}^+$ ($\mathbf{3}^-$) denotes a self-dual (antiself-dual) tensor. In $\text{SU}(2) \times \text{SU}(2)$ notation in which $\mathbf{2}_L = (\frac{1}{2}, 0)$, $\mathbf{3}^+ = (1, 0)$.

When $\epsilon = 1$ both RR and NSNS tadpoles are absent provided that $n_0 = \bar{n}_0$, $n_1 = \bar{n}_1 + 8$, and $n_2 = \bar{n}_2 + 8$. It is possible to keep only D9-branes. The generic charged massless fermions are:

$$\begin{aligned}
 & \mathbf{2}_L [(\square_0, \square_2) + (\square_1, \square_2) + (\square_0, \square_{\bar{1}}) + (\square_1, \overline{\square_2}) + k \leftrightarrow \bar{k}] + \\
 & \mathbf{2}_R [(\square_0, \square_1) + (\square_1, \overline{\square_2}) + (\square_0, \square_{\bar{2}}) + (\square_1, \square_{\bar{2}}) + k \leftrightarrow \bar{k}] + \\
 & \mathbf{2}_L \left[\begin{array}{c} \square \\ \square_1 \end{array} + \begin{array}{c} \square \\ \square_{\bar{1}} \end{array} + (\square_2, \square_{\bar{2}}) \right] + \mathbf{2}_R \left[\begin{array}{c} \square \\ \square_2 \end{array} + \begin{array}{c} \square \\ \square_{\bar{2}} \end{array} + (\square_1, \square_{\bar{1}}) \right],
 \end{aligned} \tag{3.30}$$

plus the complex conjugates. The irreducible $\text{tr } R^4$ and $\text{tr } F^4$ anomalies cancel.

With $\epsilon = -1$ the gauge groups change into $G_9 = \text{USp}(n_0) \times \text{U}(n_1) \times \text{U}(n_2)$ and $G_{\bar{9}} = \text{USp}(\bar{n}_0) \times \text{U}(\bar{n}_1) \times \text{U}(\bar{n}_2)$, where n_0 and \bar{n}_0 must be even. Tadpole cancellation requires $n_0 = \bar{n}_0$, $\bar{n}_1 = 8 + n_1$, and $\bar{n}_2 = 8 + n_2$. Massless fermions follow from (3.30) upon interchanging representations of G_9 and $G_{\bar{9}}$. The irreducible anomalies cancel as expected.

3.4.4 $D=4$, \mathbb{Z}_6^* , $v = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

We study first the Ω projection. The closed sector spectrum is:

$$\begin{aligned}
 \theta^0 & : \quad 9(0)(\text{NSNS}) + 10(0)(\text{RR}) , \\
 \theta + \theta^5 & : \quad 27(0)(\text{RR}) , \\
 \theta^2 + \theta^4 & : \quad 27 \{ 0(\text{NSNS}) + 0(\text{RR}) \} , \\
 \theta^3 & : \quad 0_{-1}(\text{NSNS}) + 10(0)(\text{RR}) .
 \end{aligned} \tag{3.31}$$

States are now labelled by helicity, 0 or $\pm \frac{1}{2}$.

As in all \mathbb{Z}_6^* examples, RR tadpoles vanish when $n_k = \bar{n}_k$, $k = 0, \dots, 3$. We also find that to cancel NSNS tadpoles, when $\epsilon = 1$, $n_0 + n_3 = 8$ and $n_1 + n_2 = 12$. The charged massless fermions are:

$$\begin{aligned}
 & (-\tfrac{1}{2}) \{ 3 [(\square_0, \overline{\square_1}) + (\square_1, \overline{\square_2}) + (\square_2, \square_3)] + [(\square_0, \square_3) + (\square_1, \square_2) + (\overline{\square_1}, \overline{\square_2})] \} + \\
 & (\tfrac{1}{2}) \{ 3 [(\square_0, \overline{\square_1}) + (\square_1, \overline{\square_2}) + (\square_2, \square_3)] + [(\square_0, \square_3) + (\square_1, \square_2) + (\overline{\square_1}, \overline{\square_2})] \} + k \leftrightarrow \bar{k} .
 \end{aligned} \tag{3.32}$$

Irreducible $\text{tr } F^3$ gauge anomalies cancel.

Let us now consider the model with Ω' projection. The closed θ^3 sector includes instead a vector plus scalars. This is:

$$\theta^3 \quad : \quad [(\pm 1) + 10(0)](\text{RR}) . \tag{3.33}$$

In the other twisted sectors tachyons are absent. Thus, this model has no closed tachyons.

Tadpoles cancel when $n_1 = 12 + \bar{n}_1$, $n_2 = 8 + \bar{n}_2$, and $n_3 = 12 + \bar{n}_3$. The massless charged fermions are:

$$\begin{aligned}
 & \left(\frac{1}{2}\right) \left\{ 3 \left[(\square_1, \bar{\square}_2) + (\square_2, \bar{\square}_3) \right] + [(\square_1, \square_3) + \text{c.c.}] + k \leftrightarrow \bar{k} \right\} + \\
 & \left(\frac{1}{2}\right) \left\{ 3 \left[\bar{\square}_1 + \bar{\square}_3 + \overline{\square\square}_1 + \overline{\square\square}_3 \right] + \left[\bar{\square}_2 + \bar{\square}_2 + \text{c.c.} \right] \right\} + \\
 & \left(-\frac{1}{2}\right) \left\{ 3 \left[(\square_1, \bar{\square}_2) + (\square_2, \bar{\square}_3) \right] + [(\square_1, \square_3) + \text{c.c.}] + k \leftrightarrow \bar{k} \right\} + \\
 & \left(-\frac{1}{2}\right) \left\{ 3 \left[(\bar{\square}_1, \bar{\square}_1) + (\square_3, \bar{\square}_3) \right] + [(\square_2, \square_2) + \text{c.c.}] \right\} .
 \end{aligned} \tag{3.34}$$

The spectrum is free of gauge anomalies. For $\bar{n}_k = 0$ it coincides with the results of [5, 6] for the non-tachyonic $0B'$ orientifold on $T^6/\mathbb{Z}_3^{\text{susy}}$. This is to be expected since here the \mathbb{Z}_6^* can be written as $\mathbb{Z}_3^{\text{susy}} \times \mathbb{Z}_2^*$, where $\mathbb{Z}_3^{\text{susy}}$ has twist vector $v = (0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, and \mathbb{Z}_2^* is $(-1)^{F_S}$.

Sector	$D=8, v = (0, 0, 0, \frac{1}{3})$	$D=6, v = (0, 0, \frac{1}{3}, \frac{2}{3})$
θ^0	$\mathbf{1} + \mathbf{1}$	$4(\mathbf{1}) + 4(\mathbf{1})$
$\theta + \theta^5$	$3 \left\{ \mathbf{1}_{-\frac{1}{3}} + [\mathbf{1} + \mathbf{15}] \right\}$	$9 \left\{ \mathbf{1}_{-\frac{1}{3}} + [\mathbf{1} + \mathbf{3}^+] \right\}$
$\theta^2 + \theta^4$	$3 \left\{ \left[\mathbf{1}_{-\frac{2}{3}} + \mathbf{1} \right] + [\mathbf{1} + \mathbf{15}] \right\}$	$9 \left\{ 4(\mathbf{1}) + [\mathbf{1} + \mathbf{3}^-] \right\}$
θ^3	$[\mathbf{1} + \mathbf{15}]$	$[2(\mathbf{1}) + \mathbf{3}^+ + 3(\mathbf{3}^-)]$
99	$U(16) \times U(16)$ $4 [(\overline{\mathbf{120}}, \mathbf{1}) + (\mathbf{1}, \mathbf{120})]$	$U(8) \times U(16) \times U(8)$ $\mathbf{2}_L [(\mathbf{8}, \mathbf{1}, \mathbf{8}) + (\overline{\mathbf{8}}, \mathbf{1}, \overline{\mathbf{8}}) + (\mathbf{1}, \mathbf{120}, \mathbf{1}) + (\mathbf{1}, \overline{\mathbf{120}}, \mathbf{1})] + \text{c.c.}$ $\mathbf{2}_R [(\mathbf{8}, \overline{\mathbf{16}}, \mathbf{1}) + (\mathbf{1}, \mathbf{16}, \overline{\mathbf{8}}) + (\mathbf{28}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{28})] + \text{c.c.}$

Table 1: Closed tachyonic and massless states, gauge group, and massless fermions in \mathbb{Z}_6^* orientifolds with Ω' projection and no $\overline{D9}$.

4. Models with O9 and O5-planes

To be concrete we study the case of $O5_1$ -planes that occurs for the \mathbb{Z}_N actions in (2.3). In general these have

$$v = \left(0, \frac{2k_1}{N}, \frac{2k_2 + 1}{N}, \frac{2k_3 + 1}{N} \right), \tag{4.1}$$

with $k_i \in \mathbb{Z}$.

4.1 Tadpole cancellation with O9 and O5-planes

It is convenient to organize tadpoles according to their volume dependence. We denote as V_1 the V_P (or V_P^{NN}) invariant under $\theta^{\frac{N}{2}}$, and as V_{23} the V_W (or V_W^{DD}) also invariant under $\theta^{\frac{N}{2}}$. As usual, the amplitudes $\mathcal{Z}_{\mathcal{K}}(\mathbb{1}, \mathbb{1})$, $\mathcal{Z}_{pp}(\mathbb{1})$ and $\mathcal{Z}_p(\mathbb{1})$, $p = 9, \bar{9}$, produce an untwisted tadpole proportional to V_{10} . Cancellation requires

$$\text{Tr } \gamma_{0,9} - \text{Tr } \gamma_{0,\bar{9}} = 32 \epsilon_9 . \tag{4.2}$$

There are also tadpoles proportional to $V_D V_1 / V_{23}$ in $D=4$, or to V_D / V_{23} in $D=6$, that arise from $\mathcal{Z}_{\mathcal{K}}(\mathbb{1}, \theta^{\frac{N}{2}})$, $\mathcal{Z}_{pp}(\mathbb{1})$ and $\mathcal{Z}_p(\theta^{\frac{N}{2}})$, $p = 5, \bar{5}$. These cancel provided

$$\text{Tr } \gamma_{0,5} - \text{Tr } \gamma_{0,\bar{5}} = 32 \delta_5 \epsilon_5, \tag{4.3}$$

where we have used (2.7) and (2.8). We will shortly explain that tadpole cancellation requires $\delta_p = -1$.

In the following we will assume the Gimon-Polchinski action for Ω , in particular, $\Omega^2 = -1$ on 95 states [15]. In our notation this means

$$\epsilon_9 = -\epsilon_5. \tag{4.4}$$

This implies for instance that in absence of $\overline{\text{D9}}$ -branes necessarily $\epsilon_5 = -1$, because (4.2) demands $\epsilon_9 = 1$.

We now discuss twisted tadpoles. Neither the Klein nor the Möbius amplitudes contribute to tadpoles due to massless states in sectors θ^m with m odd. In this case there are only cylinder tadpoles basically given by (A.22). For m odd we then have a cancellation condition

$$(\text{Tr } \gamma_{m,9} - \text{Tr } \gamma_{m,\bar{9}}) + \xi(mv) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^m)} (\text{Tr } \gamma_{m,5,J} - \text{Tr } \gamma_{m,\bar{5},J}) = 0, \tag{4.5}$$

with $J = 1, \dots, \tilde{\chi}(\theta_{\text{DD}}^m)$.

Let us now discuss tadpoles due to massless states in sectors θ^m with $m = 2\ell$ so that $T_{\mathcal{K}}^{\text{RR}}(n, \ell + k)$ and $T_p^{\text{RR}}(\ell + k)$, $n, k = 0, \frac{N}{2}$, in principle do contribute. To analyze these tadpoles it is convenient to distinguish two cases depending on whether ℓv_j , for transverse directions to the D5₁-branes, is a half-integer or not.

$\ell v_j \notin \mathbb{Z} + \frac{1}{2}, j = 2, 3.$

In this case the cylinder tadpoles add as shown in (A.22). The Klein tadpoles depend crucially on v_1 . If $\ell v_1 \in \mathbb{Z} + \frac{1}{2}$, Klein and cylinder tadpoles have a different volume dependence. Furthermore there will be uncancelled tadpoles, proportional to V_D / V_1 , because

$$\sum_{n=0, \frac{N}{2}} T_{\mathcal{K}}^{\text{RR}}(n, \ell + k) \neq 0 \quad ; \quad k = 0, \frac{N}{2}. \tag{4.6}$$

This occurs in the supersymmetric $\mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}'_8$, and \mathbb{Z}'_{12} in $D=4$ [19]. It also happens in the non-supersymmetric \mathbb{Z}_8 with $v = \frac{1}{8}(0, 2, 1, 5)$ and \mathbb{Z}_{12} with $v = \frac{1}{12}(0, 2, 1, 5)$. As advocated in [24], these problematic twisted tadpoles can be cancelled by adding pairs of D5₂- $\overline{\text{D5}}_2$ and/or D5₃- $\overline{\text{D5}}_3$ branes, besides the D9 and D5₁ required by cancellation of untwisted tadpoles.

Consider next $\ell v_1 \notin \mathbb{Z} + \frac{1}{2}$. Then, all cylinder, Möbius and Klein tadpoles have the same volume dependence $V_D V_P^{\text{NN}}$. In the appendix the relevant Möbius tadpoles from 9 and 5-branes are combined according to the sector. Comparing the Möbius result (A.30) with the cylinder tadpoles (A.22), with $m = 2\ell$, we observe that the same combination of $\text{Tr } \gamma_{2\ell,9}$ and $\text{Tr } \gamma_{2\ell,5,J}$ appears only if $\epsilon_9 = -\epsilon_5$. But this is precisely the GP condition.

Similarly, comparing (A.31) with the cylinder tadpoles we conclude that to have the same combination of traces it must be that $\delta_9 = \delta_5$.

The Klein tadpoles $T_{\mathcal{K}}^{\text{RR}}(n, \ell + k)$, $n, k = 0, \frac{N}{2}$, are directly given by (A.9) but they must be distributed among the fixed points. In particular, the contribution to the fixed point at the origin, denoted $T_{\mathcal{K},1}^{\text{RR}}(\ell)$, turns out to be

$$T_{\mathcal{K},1}^{\text{RR}}(\ell) = 2^{D+I_P} V_D V_P^{\text{NN}} \sqrt{\frac{\tilde{\chi}(\theta_{\text{NN}}^{2\ell})}{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})}} \left[\sqrt{\tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})} + \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{\ell})} \right]^2. \quad (4.7)$$

Simultaneous fixed points of θ^ℓ and $\theta^{\ell+\frac{N}{2}}$, hence also of $\theta^{\frac{N}{2}}$, receive the same contribution as that of $J = 1$. Fixed points of θ^ℓ alone get a share only from $T_{\mathcal{K}}^{\text{RR}}(0, \ell)$, this is given by the first term in (4.7) expanding the square. Similarly, fixed points of $\theta^{\ell+\frac{N}{2}}$ alone get a share only from $T_{\mathcal{K}}^{\text{RR}}(0, \ell + \frac{N}{2})$, given by the last term in (4.7) after expanding the square. Fixed points of $\theta^{2\ell}$ only do not have Klein tadpoles. Finally, when we compare the above Klein tadpole at the origin with that due to the Möbius amplitudes, c.f. (A.34), we see that necessarily $\delta_9 = -1$.

We are now ready to add the cylinder, Klein and Möbius tadpoles. The above results show that they add to a common prefactor times a perfect square. Including antibranes we then find the cancellation condition

$$\begin{aligned} & (\text{Tr } \gamma_{2\ell,9} - \text{Tr } \gamma_{2\ell,\bar{9}}) + \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})} (\text{Tr } \gamma_{2\ell,5,J} - \text{Tr } \gamma_{2\ell,\bar{5},J}) = \\ & 2^{\frac{1}{2}(D+I_P)} \epsilon_9 \prod_{a=0}^3 c(\ell v_a) \left[\sqrt{\tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})} + \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{\ell})} \right], \end{aligned} \quad (4.8)$$

where I_P is the dimension of the momentum sub-lattice invariant under $\theta^{2\ell}$. This is valid when J refers to a simultaneous fixed point of θ^ℓ and $\theta^{\ell+\frac{N}{2}}$, such as the origin. For fixed points of θ^ℓ alone only the first term in the right hand side of (4.8) appears. For fixed points of $\theta^{\ell+\frac{N}{2}}$ is only the second term that appears. For fixed points of $\theta^{2\ell}$ alone the right hand side is zero.

When N is a multiple of four we can consider $\ell = \frac{N}{4}$. Then, the above results apply if $\frac{N}{4}v_1 \notin \mathbb{Z} + \frac{1}{2}$. In this case we can check that the quantity inside brackets in (4.8) vanishes identically. We then find the tadpole cancellation condition

$$\left(\text{Tr } \gamma_{\frac{N}{2},9} - \text{Tr } \gamma_{\frac{N}{2},\bar{9}} \right) - 4 \left(\text{Tr } \gamma_{\frac{N}{2},5,J} - \text{Tr } \gamma_{\frac{N}{2},\bar{5},J} \right) = 0 \quad ; \quad J = 1, \dots, 16. \quad (4.9)$$

Examples are given by the \mathbb{Z}_4 in $D=6$, supersymmetric or not. Recall however that if $\frac{N}{4}v_1 \in \mathbb{Z} + \frac{1}{2}$, the Klein and cylinder tadpoles have a different volume dependence and moreover there will be uncanceled tadpoles, proportional to V_D/V_1 , as in the supersymmetric \mathbb{Z}_4 in $D=4$ [19].

$\ell v_j \in \mathbb{Z} + \frac{1}{2}$, for $j = 2$ and/or $j = 3$.

In this case it happens that all RR tadpoles from $\mathcal{Z}_{95}(\theta^{2\ell})$, $\mathcal{Z}_9(\theta^\ell)$, $\mathcal{Z}_5(\theta^{\ell+\frac{N}{2}})$, $\mathcal{Z}_{\mathcal{K}}(\theta^{\frac{N}{2}}, \theta^\ell)$, and $\mathcal{Z}_{\mathcal{K}}(\theta^{\frac{N}{2}}, \theta^{\ell+\frac{N}{2}})$, do vanish. Furthermore, the tadpoles $T_{99}^{\text{RR}}(2\ell)$, $T_9^{\text{RR}}(\ell + \frac{N}{2})$ and $T_{\mathcal{K}}^{\text{RR}}(0, \ell + \frac{N}{2})$, all have the same volume dependence $V_D V_P$. Adding all contributions gives the cancellation condition

$$\text{Tr } \gamma_{2\ell,9} - \text{Tr } \gamma_{2\ell,\bar{9}} = 2^{\frac{1}{2}(D+I_P)} \epsilon_9 \delta_9 \prod_{a=0}^3 c(\ell v_a + \frac{N}{2} v_a) , \quad (4.10)$$

where I_P is the dimension of the momentum sub-lattice invariant under $\theta^{2\ell}$. To obtain this result we have assumed that $\tilde{\chi}(\theta_{\text{DD}}^{2\ell}) = \tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})$ which does hold for the crystallographic actions. Notice that for $\ell = \frac{N}{2}$ the above cancellation condition reproduces (4.2).

The remaining tadpoles $T_{55}^{\text{RR}}(2\ell)$, $T_5^{\text{RR}}(\ell)$ and $T_{\mathcal{K}}^{\text{RR}}(0, \ell)$ all have the same volume dependence $V_D V_P^{\text{NN}}/V_W^{\text{DD}}$. The way these pieces combine depends on the dimension of the invariant subspace of $\theta^{2\ell}$ in the DD directions, When $\theta^{2\ell}$ leaves invariant the whole sub-space transverse to the 5-branes the tadpole cancellation condition turns out to be

$$\text{Tr } \gamma_{2\ell,5} - \text{Tr } \gamma_{2\ell,\bar{5}} = 2^{\frac{1}{2}(D+I_P)} \epsilon_5 \prod_{a=0}^3 c(\ell v_a + \frac{N}{2} v_a) . \quad (4.11)$$

As a check, observe that for $\ell = \frac{N}{2}$ we recover the condition (4.3). The remaining possibility is that $\theta^{2\ell}$ leaves invariant only one of the transverse complex directions, say Y^3 . Factorization requires that θ^ℓ leaves fixed only the origin ($J = 1$) in the Y^2 direction, and this is true for crystallographic actions. We then find a cancellation condition

$$\text{Tr } \gamma_{2\ell,5,J} - \text{Tr } \gamma_{2\ell,\bar{5},J} = 2^{\frac{1}{2}(D+I_P)} \epsilon_5 \prod_{a=0}^3 c(\ell v_a + \frac{N}{2} v_a) \quad ; \quad J = 1 . \quad (4.12)$$

For other fixed points of $\theta^{2\ell}$ in the Y^2 direction there is an analogous condition but with the right hand side equal to zero.

When all D5-branes are located at the origin, T-duality imposes that $\text{Tr } \gamma_{2\ell,9} = \text{Tr } \gamma_{2\ell,5}$. Thus, $\epsilon_9 \delta_9 = \epsilon_5$ and the GP condition implies $\delta_9 = -1$. Repeating the argument with $\ell \rightarrow \ell + \frac{N}{2}$ shows that $\delta_5 = -1$.

4.2 Examples

The tadpole cancellation conditions that we have derived apply to all \mathbb{Z}_N orientifolds with action of the form (4.1). These include our non-supersymmetric as well as supersymmetric orientifolds considered previously [15–20]. Our results apply when the D5-branes are located at generic fixed points of θ or some power. We will exemplify the solutions of the tadpole conditions, and the corresponding spectra, only for a few cases but we will consider moving the D5-branes to the bulk or to other fixed points different from the origin.

We will first analyze three \mathbb{Z}_6 orientifolds in which the distinct possibilities for even twisted tadpoles arise. We then study a \mathbb{Z}_8 orientifold in order to further illustrate the importance of having tadpole cancellation conditions that take into account all fixed points and not only the origin.

In general, $\gamma_{1,9}$ and $\gamma_{1,5,J}$ are of the form (3.18), and analogous for antibranes. For Ω we take the realization given in (3.20). Recall that we are assuming the GP action $\epsilon_5 = -\epsilon_9$. We will explicitly discuss examples with $\epsilon_9 = 1$. We have found that the opposite orientifold projection, $\epsilon_9 = -1$, simply leads to the replacement of branes by antibranes and, whenever they appear, of USp by SO groups.

4.2.1 $D=6$, $v = (0, 0, \frac{1}{6}, \frac{5}{6})$

For this \mathbb{Z}_6 action we take the torus lattice to be the product of two SU(3) root lattices. The only fixed point of θ is the origin denoted X_1 . The fixed points of θ^2 are of the form (w_i, w_j) , where $w_0 = (0, 0)$, whereas w_1 and w_2 are the SU(3) weights. Thus, besides the origin X_1 , θ^2 has eight fixed points labelled X_J . Notice also that θ^3 has sixteen fixed points. The chiral massless fields coming from the closed sector are shown in table 2.

Twist	θ^0	θ, θ^5	θ^2, θ^4	θ^3
$v = (0, 0, \frac{1}{6}, \frac{5}{6})$	–	$\mathbf{3}^+ + 6(\mathbf{2}_L)$	$5(\mathbf{3}^-) + 16(\mathbf{2}_R)$	$10(\mathbf{2}_L)$
$v = (0, 0, \frac{1}{6}, \frac{1}{2})$	$2(\mathbf{2}_R)$	$4(\mathbf{3}^+) + 8(\mathbf{2}_L)$	$2(\mathbf{3}^+) + 2(\mathbf{3}^-) + 4(\mathbf{2}_L) + 2(\mathbf{2}_R)$	$8(\mathbf{2}_R)$

Table 2: Chiral massless tensors and fermions in closed sectors of $D=6$ \mathbb{Z}_6 orientifolds.

The model-dependent twisted tadpole cancellation conditions are obtained from (4.5) for $m = 1, 3$, and from (4.8) with $m = 2$, and distinguishing among the fixed points of θ^2 . We find

$$\begin{aligned}
 (\mathrm{Tr} \gamma_{1,9} - \mathrm{Tr} \gamma_{1,\bar{9}}) - (\mathrm{Tr} \gamma_{1,5,1} - \mathrm{Tr} \gamma_{1,\bar{5},1}) &= 0, \\
 (\mathrm{Tr} \gamma_{2,9} - \mathrm{Tr} \gamma_{2,\bar{9}}) + 3(\mathrm{Tr} \gamma_{2,5,1} - \mathrm{Tr} \gamma_{2,\bar{5},1}) &= -32\epsilon_9, \\
 (\mathrm{Tr} \gamma_{2,9} - \mathrm{Tr} \gamma_{2,\bar{9}}) + 3(\mathrm{Tr} \gamma_{2,5,J} - \mathrm{Tr} \gamma_{2,\bar{5},J}) &= -8\epsilon_9 \quad ; \quad J = 2, \dots, 9, \\
 (\mathrm{Tr} \gamma_{3,9} - \mathrm{Tr} \gamma_{3,\bar{9}}) - 4(\mathrm{Tr} \gamma_{3,5,K} - \mathrm{Tr} \gamma_{3,\bar{5},K}) &= 0 \quad ; \quad K = 1, \dots, 16.
 \end{aligned}
 \tag{4.13}$$

For the supersymmetric $D=6$, \mathbb{Z}_6 orientifold with $v = (0, 0, \frac{1}{6}, -\frac{1}{6})$, there arise similar conditions as first derived in [16, 17]. However, in previous works the condition involving fixed points of θ^2 other than the origin, is often missed. As stressed in [21], such conditions are needed to ensure anomaly cancellation.

We fix $\epsilon_9 = 1$ and focus on solutions without antibranes. We then deduce that there must always be a number of D5-branes sitting at the origin. It also follows that D5-branes can be placed at either all of the X_J or none of them. In the case without branes at the X_J we can further move $12k$ D5-branes to the bulk, say $2k$ to a point X and the remaining to the images of X under θ . In this situation the generic gauge group turns out to be

$$\begin{aligned}
 G_9 &= \mathrm{U}(n) \times \mathrm{U}(8-n) \times \mathrm{U}(8), \\
 G_{5_1} &= \mathrm{U}(n-2k) \times \mathrm{U}(8-n-2k) \times \mathrm{U}(8-2k), \\
 G_{5_X} &= \mathrm{USp}(2k),
 \end{aligned}
 \tag{4.14}$$

where $2k \leq n \leq (8-2k)$. Notice that only $k = 0, 1, 2$, are allowed.

The massless fermions in each sector are given in table 3. It is easy to check that the irreducible $\text{tr } F^4$ anomaly cancels for each group factor. For this it is crucial that the 95 fermions have left chirality as required by the GSO projection explained in section 2.3.3. The gravitational anomaly also cancels since eq. (3.1) is verified when we take into account the closed string spectrum shown in table 2.

Sector	massless fermions
99, $5_1 5_1$	$\mathbf{2}_L: (\square, \square, \mathbf{1}) + (\bar{\square}, \bar{\square}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \square) + (\mathbf{1}, \mathbf{1}, \bar{\square}) + \text{c.c.}$ $\mathbf{2}_R: (\square, \bar{\square}, \mathbf{1}) + (\square, \mathbf{1}, \square) + (\mathbf{1}, \bar{\square}, \bar{\square}) + \text{c.c.}$
95_1	$\mathbf{2}_L: (\square, \mathbf{1}, \mathbf{1}; \bar{\square}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \square, \mathbf{1}; \mathbf{1}, \bar{\square}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \square; \mathbf{1}, \mathbf{1}, \bar{\square}) + \text{c.c.}$
$5_X 5_X$	$\mathbf{2}_L: 2 \times \square \quad ; \quad \mathbf{2}_R: 2 \times \square$
95_X	$\mathbf{2}_L: (\square, \mathbf{1}, \mathbf{1}; \square) + (\mathbf{1}, \square, \mathbf{1}; \square) + (\mathbf{1}, \mathbf{1}, \square; \square) + \text{c.c.}$

Table 3: Massless fermions in open sectors of the \mathbb{Z}_6 , $v = (0, 0, \frac{1}{6}, \frac{5}{6})$, orientifold.

We can also move branes from the origin to X_J fixed points. There can be configurations with sixteen branes left at X_1 and the remaining distributed among the X_J . For example, there is a solution with

$$\begin{aligned}
 G_9 &= \text{U}(n) \times \text{U}(10 - n) \times \text{U}(6) , \\
 G_{5_1} &= \text{U}(n - 4) \times \text{U}(6 - n) \times \text{U}(6) , \\
 G_{5_J} &= \text{SU}(2)^4 .
 \end{aligned}
 \tag{4.15}$$

Clearly only $n = 4, 5, 6$, are allowed. In this solution there are two D5-branes at each of the eight X_J , but the gauge group is only $\text{SU}(2)^4$ because these fixed points form doublets under θ . The massless fermions in the 99, $5_1 5_1$ and 95_1 sectors are the same, *mutatis mutandis*, as those in table 3. In the $5_I 5_I$ sector, compared to the $5_X 5_X$ in table 3, we have to be careful that now there is a θ^2 projection that removes the symmetric representation and leaves the antisymmetric that is a singlet in this case. Similarly, in the 95_I sector the θ^2 projection only allows $\mathbf{2}_L$ massless fermions transforming as

$$(\mathbf{1}, \mathbf{1}, \square; \square) + \text{c.c.}
 \tag{4.16}$$

for each $\text{SU}(2)$. The full massless spectrum is anomaly-free as expected.

There is another solution with two D5 at each X_J in which $G_{5_J} = \text{U}(1)^4$, but we skip further details. It is also possible to move D5-branes from the origin to the fixed points of θ^3 .

4.2.2 $D=6$, $v = (0, 0, \frac{1}{6}, \frac{1}{2})$

The torus lattice is an $\text{SU}(3)$ times an $\text{SO}(4)$ root lattice. The element θ has four fixed points of type $(0, 0) \otimes (\frac{a}{2}, \frac{b}{2})$, with $a, b = 0, 1$. The element θ^2 leaves the complex direction Y^3 fixed, whereas in the direction Y^2 the fixed points are the origin and the weights w_1 and w_2 of $\text{SU}(3)$. The closed massless chiral states are displayed in table 2.

The cancellation conditions for odd twisted tadpoles follow from (4.5) but in this model the θ^2 tadpoles follow instead from (4.10) and (4.12). Explicitly they read

$$\begin{aligned}
 (\text{Tr } \gamma_{1,9} - \text{Tr } \gamma_{1,\bar{9}}) + 2 (\text{Tr } \gamma_{1,5,J} - \text{Tr } \gamma_{1,\bar{5},J}) &= 0 \quad ; \quad J = 1, \dots, 4, \\
 (\text{Tr } \gamma_{2,9} - \text{Tr } \gamma_{2,\bar{9}}) &= (\text{Tr } \gamma_{2,5,1} - \text{Tr } \gamma_{2,\bar{5},1}) = 16\epsilon_9, \\
 (\text{Tr } \gamma_{2,5,L} - \text{Tr } \gamma_{2,\bar{5},L}) &= 0 \quad ; \quad L = 2, 3.
 \end{aligned}
 \tag{4.17}$$

The condition involving $\gamma_{3,p}$ is automatically satisfied with our choice (3.18).

We set $\epsilon_9 = 1$ and declare that $\overline{\text{D9}}$ -branes are absent. Besides, in general there is a net number of 32 D5-branes plus in principle some D5- $\overline{\text{D5}}$ pairs. In fact, if we try to move $12k$ branes from the origin to the bulk these pairs will be manifest. Observe that all fixed points of θ are equivalent but we choose to refer to the origin. The solution of the tadpole cancellation conditions with minimum number of D5- $\overline{\text{D5}}$ pairs at the origin leads to gauge groups

$$\begin{aligned}
 G_9 &= \text{U}(8) \times \text{U}(8), \\
 G_{5_1} &= \text{U}(8 - 2k) \times \text{U}(8 - 2k) \quad ; \quad G_{\bar{5}_1} = \text{U}(2k), \\
 G_{5_X} &= \text{USp}(2k).
 \end{aligned}
 \tag{4.18}$$

This solution has the property that tachyons from $5_1\bar{5}_1$ strings are eliminated by the orbifold projection. Massless fermions from $\bar{5}_1\bar{5}_1$ and $9\bar{5}_1$ strings are projected out too. The spectrum of surviving massless charged fermions is presented in table 4. It is easy to prove that irreducible gauge and gravitational anomalies cancel.

Sector	massless fermions
99, $5_1\bar{5}_1$	$\mathbf{2}_L: (\square, \bar{\square}) + \text{c.c.}$ $\mathbf{2}_R: \left(\begin{smallmatrix} \square \\ \square \end{smallmatrix}, \mathbf{1}\right) + \left(\mathbf{1}, \begin{smallmatrix} \square \\ \square \end{smallmatrix}\right) + \text{c.c.}$
95 ₁	$\mathbf{2}_R: (\square, \mathbf{1}; \bar{\square}, \mathbf{1}) + (\mathbf{1}, \square; \mathbf{1}, \bar{\square}) + \text{c.c.}$
$5_1\bar{5}_1$	$\mathbf{2}_L: (\square, \mathbf{1}; \bar{\square}) + (\mathbf{1}, \square; \bar{\square}) + \text{c.c.}$ $\mathbf{2}_R: (\square, \mathbf{1}; \square) + (\mathbf{1}, \square; \square) + \text{c.c.}$
$5_X\bar{5}_X$	$\mathbf{2}_L: 2 \times \begin{smallmatrix} \square \\ \square \end{smallmatrix} \quad ; \quad \mathbf{2}_R: 2 \times \square\square$
95 _X	$\mathbf{2}_L: (\square, \mathbf{1}; \square) + (\mathbf{1}, \square; \square) + \text{c.c.}$

Table 4: Massless fermions in open sectors of the \mathbb{Z}_6 , $v = (0, 0, \frac{1}{6}, \frac{1}{2})$, orientifold.

Instead of moving D5-branes from the origin, or actually from any of the θ fixed points, to the bulk, we can place them in groups of $6k$ at the two fixed points, $L = 2, 3$, of θ^2 (in the Y^2 direction) that are exchanged by θ . These D5-branes give rise to the group

$$G_{5_L} = \text{U}(2k) \times \text{USp}(2k). \tag{4.19}$$

The USp factor appears because the D5-branes sit at fixed tori, since the element θ^2 leaves the Y^3 direction invariant. The states from D9, D5₁ and $\overline{\text{D5}}_1$ are still those given in table 4. The massless fermions from the D5_L turn out to be:

$$\begin{aligned}
 5_L 5_L \mathbf{2}_L &: (\square, \mathbf{1}) + (\square; \square) + \text{c.c.} \\
 \mathbf{2}_R &: \left(\begin{array}{c} \square \\ \square \end{array}; \mathbf{1} \right) + (\square, \square) + \text{c.c.} \\
 95_L \mathbf{2}_R &: (\square, \mathbf{1}; \overline{\square}, \mathbf{1}) + (\mathbf{1}, \square; \square, \mathbf{1}) + \text{c.c.} .
 \end{aligned} \tag{4.20}$$

We have verified that the full content is anomaly-free.

Finally, there is a configuration with no $\overline{\text{D5}}$ and the 32 D5 equally distributed at the four fixed points of θ . The gauge groups in this case are

$$G_9 = \text{U}(12 - 2n) \times \text{U}(4 + 2n) \quad ; \quad G_{5_J} = \text{U}(n) \times \text{U}(4 - n) \quad ; \quad J = 1, \dots, 4 . \tag{4.21}$$

Notice that the rank of the group arising from D5-branes is sixteen. The representations for the massless fermions coincide with those given in table 4 for 99, 5₁5₁, and 95₁.

4.2.3 $D=4$, $v = (0, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$

This \mathbb{Z}_6 rotation can be realized on the product of $\text{SU}(3) \times \text{SO}(4) \times \text{SO}(4)$ root lattices. Since the D5-branes wrap the first sub-torus, only the sixteen fixed points X_J of θ in the transverse complex directions (Y^2, Y^3) are relevant. The tadpole cancellation conditions turn out to be

$$\begin{aligned}
 (\text{Tr } \gamma_{m,9} - \text{Tr } \gamma_{m,\overline{9}}) + 4 (\text{Tr } \gamma_{m,5,J} - \text{Tr } \gamma_{m,\overline{5},J}) &= 0 \quad ; \quad m = 1, 3 \quad ; \quad J = 1, \dots, 16 , \\
 (\text{Tr } \gamma_{2,9} - \text{Tr } \gamma_{2,\overline{9}}) = (\text{Tr } \gamma_{2,5} - \text{Tr } \gamma_{2,\overline{5}}) &= 16\epsilon_9 .
 \end{aligned} \tag{4.22}$$

We consider $\epsilon_9 = 1$ and assume that there are only 32 D9-branes present. For D5-branes we allow some number of D5- $\overline{\text{D5}}$ pairs.

There are tadpole-free solutions with only 32 D5-branes distributed among the fixed points of θ . For example, we can locate two D5-branes at each X_J , thereby obtaining a model with gauge groups $G_9 = \text{U}(10) \times \text{U}(6)$ and $G_5 = \text{U}(1)^{16}$. Now, when we try to move D5-branes to the bulk some D5- $\overline{\text{D5}}$ pairs are forced to remain at fixed points. To see this, we first place 32 D5-branes at some specific X_J and then displace $2k$ D5-branes, plus their $2k$ θ -images, to the bulk that is fixed by θ^2 . We then find that necessarily $4k$ $\overline{\text{D5}}$ -branes must remain at X_J . The setup with least number of $\overline{\text{D5}}$'s has gauge group

$$\begin{aligned}
 G_9 &= \text{U}(8) \times \text{U}(8) , \\
 G_{5_J} &= \text{U}(8) \times \text{U}(8) \quad ; \quad G_{\overline{5}_J} = \text{U}(2k) , \\
 G_{5_X} &= \text{USp}(2k) .
 \end{aligned} \tag{4.23}$$

Tachyons from $5_J \overline{5}_J$ and massless fermions from $\overline{5}_J \overline{5}_J$ strings are removed by the orbifold projection. The spectrum of massless charged fermions is presented in table 5. It is not difficult to check that $\text{tr } F^3$ anomalies cancel.

Sector	massless fermions
$99, 5_J 5_J$	$(\frac{1}{2}): 2 \times [(\square, \overline{\square}) + (\overline{\square}, \mathbf{1}) + (\mathbf{1}, \overline{\square})]$
95_J	$(\frac{1}{2}): (\overline{\square}, \mathbf{1}; \overline{\square}, \mathbf{1}) + (\mathbf{1}, \square; \mathbf{1}, \square)$
$9\overline{5}_J$	$(-\frac{1}{2}): (\square, \mathbf{1}; \overline{\square}) + (\mathbf{1}, \overline{\square}; \square)$
$5_J \overline{5}_J$	$(-\frac{1}{2}): 2 \times [(\square, \mathbf{1}; \overline{\square}) + (\mathbf{1}, \overline{\square}; \square) + (\overline{\square}, \mathbf{1}; \overline{\square}) + (\mathbf{1}, \square; \square)]$
95_X	$(\frac{1}{2}): (\square, \mathbf{1}; \square) + (\mathbf{1}, \overline{\square}; \square)$

Table 5: Massless fermions in open sectors of the \mathbb{Z}_6 , $v = (0, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, orientifold.

4.2.4 $D=6$, $v = (0, 0, \frac{1}{8}, \frac{3}{8})$

This \mathbb{Z}_8 rotation allows a realization on a 4-dimensional hypercubic lattice with orthonormal basis e_i . The action is $\theta e_i = e_{i+1}$, $i = 1, 2, 3$, and $\theta e_4 = -e_1$. The fixed points of θ , denoted X_1 and X_2 , are respectively the origin and the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ at the center of the fundamental lattice cell. The element θ^2 has four fixed points, the two above plus X_3 and X_4 given by $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$. The element θ^4 has sixteen fixed points, the four X_J that are singlets under θ^2 , plus another twelve points that form doublets of θ^2 .

The closed string spectrum can be found using the results of ref. [22] and implementing the Ω projection. We find that there is a net number of three tensors $\mathbf{3}^+$, but equal numbers of $\mathbf{2}_L$ and $\mathbf{2}_R$. Thus, there is gravitational anomaly that must be compensated by open string massless fermions.

The significant tadpole cancellation equations are obtained from our general analysis. They read:

$$\begin{aligned}
 \text{Tr } \gamma_{1,9} + \sqrt{2} \text{Tr } \gamma_{1,5,J} &= 0 \quad ; \quad J = 1, 2, \\
 \text{Tr } \gamma_{2,9} + 2 \text{Tr } \gamma_{2,5,J} &= 16\sqrt{2} \quad ; \quad J = 1, 2, \\
 \text{Tr } \gamma_{2,9} + 2 \text{Tr } \gamma_{2,5,J} &= 0 \quad ; \quad J = 3, 4, \\
 \text{Tr } \gamma_{3,9} - \sqrt{2} \text{Tr } \gamma_{3,5,J} &= 0 \quad ; \quad J = 1, 2.
 \end{aligned}
 \tag{4.24}$$

Here we have already set $\epsilon_9 = 1$ and assumed that antibranes are absent.

The conditions involving the origin agree with general results in ref. [20] that however do not take other fixed points into account. Now, neglecting cancellation conditions at other fixed points leads to the wrong outcome that all 32 D5-branes could be placed at the origin. Indeed, it is simple to prove that such configuration is not anomaly-free. In our approach this is evident from the fact that the equations (4.24) imply that if there are D5-branes located at X_1 then necessarily there is the same number at X_2 . We can have a generic arrangement with $(16 - 2k)$ D5-branes at each X_1 and X_2 , together with $2k$ at each X_3 and X_4 .

As an example, with D5-branes only at X_1 and X_2 , the tadpole-free solution for the $\gamma_{m,p}$ matrices yields gauge groups $G_9 = \text{U}(j)^2 \times \text{U}(8-j)^2$ and $G_{5_1} = G_{5_2} = \text{U}(8-j) \times \text{U}(j)$.

We refrain from displaying the full list of massless fermions that is straightforward to derive using our prescriptions. To make the main observations it is enough to consider the simplest case $j = 0$. In this case the massless fermions are just:

$$\begin{aligned}
 99 \quad \mathbf{2}_L &: (\mathbf{8}, \mathbf{8}) + \text{c.c.} \\
 \mathbf{2}_R &: (\mathbf{8}, \bar{\mathbf{8}}) + (\mathbf{1}, \mathbf{28}) + \text{c.c.} \\
 5_1 5_1, 5_2 5_2 \quad \mathbf{2}_R &: \mathbf{28} + \text{c.c.} .
 \end{aligned}
 \tag{4.25}$$

Clearly, $N_R - N_L = 84$, as needed to cancel gravitational anomalies. The $\text{tr } F^4$ anomalies vanish as well. Note that in general $G_9 = G_{5_1} \times G_{5_2}$ so that we might say that the brane configuration is T-dual. However, as the $j = 0$ example shows, the matter content is not 95 symmetric.

Another simple solution has sixteen D5-branes at each X_3 and X_4 . Since these points are exchanged by θ , there is only a gauge factor G_5 of rank eight, in fact $G_5 = U(8)$. We also find $G_9 = U(8) \times U(8)$ and massless fermions

$$\begin{aligned}
 99 \quad \mathbf{2}_R &: (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{28}) + \text{c.c.} \\
 55 \quad \mathbf{2}_R &: \mathbf{28} + \text{c.c.} .
 \end{aligned}
 \tag{4.26}$$

We can argue that this model is connected to that in eq. (4.25) upon T-duality in the compact directions so that $D9 \leftrightarrow D5$. Clearly, the group $G_{5_1} \times G_{5_2}$ and its matter content match the new 99 sector. Concerning the starting 99 sector, the idea is that we turn on Wilson lines to separate branes in two stacks of sixteen each so that the bifundamentals become massive. To reduce the rank and keep \mathbb{Z}_8 invariance these stacks must be at fixed points of θ^2 .

5. Models with O9-planes and O7 or O3-planes

The prototypical examples having O7 and O3-planes are furnished by the orientifolds with \mathbb{Z}_4 action given by the twist vectors $v = (0, 0, 0, \frac{1}{2})$ and $v = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ respectively. These non-supersymmetric orientifolds have been discussed in refs. [7, 9]. Here we will basically reproduce their results using our formalism. In particular, in contrast to [7, 9], in our language from the beginning it is clear that a generator θ that just reflects an odd number of complex coordinates is truly of order four. This is required by modular invariance and it is consistent with the fact that such θ has order four acting on world-sheet fermions. In this section we mainly discuss the \mathbb{Z}_4 examples but the same procedure can be applied to more general cases, for example when $Nv = (0, 4k_1, 4k_2, 4k_3 + 2)$, or $Nv = (0, 4k_1 + 2, 4k_2 + 2, 4k_3 + 2)$, $k_i \in \mathbb{Z}$, and N is multiple of four.

A distinguishing property of this class of models is that $\Omega^2 = (-1)^{F_S}$, when acting on pp strings, $p = 7, 3$, as can be shown by looking how Ω acts on states in the Ramond sector. In turn this property implies that

$$\gamma_{\Omega,p} (\gamma_{\Omega,p}^T)^{-1} = \epsilon_p \gamma_{N/2,p},
 \tag{5.1}$$

where we have used that $\theta^{N/2} = (-1)^{F_S}$. It then follows that $\epsilon_p^4 = 1$. On the other hand, acting on $9p$ states we have the GP action $\Omega^2 = e^{i\pi(9-p)/4}$.

To discuss tadpole cancellation conditions we will center on the T^{9-p}/\mathbb{Z}_4 orientifolds, with $p = 7$ or $p = 3$, that have O9-planes together with Op -planes. This is enough to illustrate the main features. To start we analyze the divergences in the Klein-bottle amplitude. Tadpoles originate from $\mathcal{Z}_{\mathcal{K}}(\mathbb{1}, \theta^m)$ and $\mathcal{Z}_{\mathcal{K}}(\theta^{\frac{N}{2}}, \theta^m)$, for $m = 0, 2$, they are proportional to V_{10} , whereas for $m = 1, 3$, they depend on V_D/V_W . When $\Omega_P = \Omega$, the RR tadpoles actually vanish, i.e. eq. (3.3) also holds in this situation. Contrariwise, when $\Omega_P = \Omega'$ the divergences from the untwisted and the $\frac{N}{2}$ -twisted sector combine to yield a non-zero RR tadpole and D9 plus Dp -branes must be added. This is the most interesting case and we will exclusively focus on it in the following.

Tadpoles due to cylinder amplitudes \mathcal{Z}_{pp} are given by eq. (A.18). Concerning \mathcal{Z}_{p9} , it corresponds to the amplitude of a sector twisted by v_{p9} with entries $\frac{1}{2}$ in the DN directions. We can take $v_{p9} = \frac{N}{4}v$ for $p = 3, 7$. Then there is a modified spin structure $s_{0\frac{1}{2}} = -e^{-i\pi N S_v/4}$ that enters in RR tadpoles and must be taken into account in the GSO projection of $p9$ states. It also happens that if Y^j is DN and $mv_j \in \mathbb{Z}$, the RR tadpole from $\mathcal{Z}_{p9}(\theta^m)$ is zero. Notice that the DN (ND) coordinates for $p9$ ($9p$) strings are the same as the DD for pp strings. If $mv_j \notin \mathbb{Z}$ for the DD coordinates, all cylinder tadpoles have the same volume dependence and we have a result analogous to (A.22) with the replacement

$$\xi(mv) \rightarrow \hat{\xi}(mv) = -e^{-i\pi \frac{N}{4} S_v} \prod_{\text{DD}} s(mv_j), \quad (5.2)$$

because the number of DD directions is now odd and there is a different v_{p9} .

The Möbius amplitudes for D7 and D3-branes are also similar to those for D5-branes explained in the appendix. In fact, the Möbius tadpoles can be read off from the formulas in section A upon obvious modifications.

Putting together all necessary results leads to tadpole cancellation conditions that can be organized according to volume dependence. The odd twisted tadpoles only receive contributions from cylinder amplitudes and are proportional to V_D . For instance, in the \mathbb{Z}_4 in $D=8$ we find

$$\begin{aligned} \text{Tr } \gamma_{1,9} + 2i \text{Tr } \gamma_{1,7,J} &= 0 \quad ; \quad J = 1, \dots, 4, \\ \text{Tr } \gamma_{3,9} - 2i \text{Tr } \gamma_{3,7,J} &= 0. \end{aligned} \quad (5.3)$$

To simplify we are not including antibranes.

Tadpoles proportional to V_{10} give the condition

$$(\text{Tr } \gamma_{0,9})^2 + (\text{Tr } \gamma_{2,9})^2 - 2^6 \epsilon_9 (1 - \delta_9) \text{Tr } \gamma_{0,9} + 2^{12} = 0, \quad (5.4)$$

where we have used (2.8) valid for D9-branes. The solution must be $\text{Tr } \gamma_{2,9} = 0$, $\delta_9 = -1$ and

$$\text{Tr } \gamma_{0,9} = 64\epsilon_9. \quad (5.5)$$

Since antibranes are absent, necessarily $\epsilon_9 = 1$ and there are 64 D9-branes.

Cancellation of the remaining tadpoles proportional to V_D/V_W involves the D7, or D3-branes, and requires instead

$$(\text{Tr } \gamma_{0,p})^2 + (\text{Tr } \gamma_{2,p})^2 + 2^6 \left[\text{Tr } \gamma_{\Omega 1,p}^{-1} \gamma_{\Omega 1,p}^T - \text{Tr } \gamma_{\Omega 3,p}^{-1} \gamma_{\Omega 3,p}^T \right] + 2^{12} = 0. \quad (5.6)$$

The solution now is $\text{Tr } \gamma_{2,p} = 0$ together with

$$\gamma_{\Omega_{1,p}}^{-1} \gamma_{\Omega_{1,p}}^T = -\gamma_{0,p} \quad ; \quad \gamma_{\Omega_{3,p}}^{-1} \gamma_{\Omega_{3,p}}^T = \gamma_{0,p}, \quad (5.7)$$

so that $\text{Tr } \gamma_{0,p} = 64$.

We will now present the matrices that fulfill the above conditions. Consider first the D9 sector. Since $\gamma_{1,9}^4 = -\mathbb{1}$, and there are 64 D9-branes, we can realize θ as

$$\gamma_{1,9} = \text{diag}(\nu \mathbb{1}_{16}, \nu^3 \mathbb{1}_{16}, \bar{\nu}^3 \mathbb{1}_{16}, \bar{\nu} \mathbb{1}_{16}) \quad ; \quad \nu = e^{i\pi/4}. \quad (5.8)$$

Notice that this is eq. (3.18) with $n_j = 16$. Clearly, $\text{Tr } \gamma_{1,9} = \text{Tr } \gamma_{2,9} = 0$. The world sheet parity Ω' is represented by $\gamma_{\Omega,9}^+$ given in eq. (3.20).

In the Dp sector we will locate all 64 branes at the origin so that we can choose $\gamma_{1,p} = \gamma_{1,9}$. The orientifold action is instead given by

$$\gamma_{\Omega,p} = \begin{pmatrix} 0 & 0 & 0 & \nu \mathbb{1}_{16} \\ 0 & 0 & \bar{\nu} \mathbb{1}_{16} & 0 \\ 0 & \nu \mathbb{1}_{16} & 0 & 0 \\ \bar{\nu} \mathbb{1}_{16} & 0 & 0 & 0 \end{pmatrix}. \quad (5.9)$$

Therefore, $\gamma_{\Omega m,p} = \gamma_{m,p} \gamma_{\Omega,p}$ satisfies the conditions (5.7). Moreover,

$$\gamma_{\Omega,p} (\gamma_{\Omega,p}^T)^{-1} = \gamma_{2,p}. \quad (5.10)$$

Comparing with the general property (5.1) we see that $\epsilon_p = 1$. Besides, notice that $\gamma_{1,p}^4 = -\mathbb{1}$.

Let us now study the massless open spectrum. The gauge groups are easy to determine, we find $G_9 = G_p = \text{U}(16) \times \text{U}(16)$. In the $D=4$ orientifold with D9 and D3-branes, fermions are labelled by their chirality $\pm \frac{1}{2}$. To be concrete we look at this case and later state the results for D7-branes. Using the general rules explained in section 2.3.1 we readily find that the 99 massless fermions are

$$4\left(\frac{1}{2}\right) [(\mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{120}) + (\overline{\mathbf{16}}, \overline{\mathbf{16}})] . \quad (5.11)$$

For 33 massless fermions we have to be careful that the action of Ω on the Ramond sector includes a factor of i , that implies $\Omega^2 = (-1)^{F_S}$ as mentioned before. The \mathbb{Z}_4 and Ω projection on 33 fermions of positive chirality require a fermionic Chan-Paton factor satisfying

$$\lambda_F = i\gamma_{1,3} \lambda_F \gamma_{1,3}^{-1} \quad ; \quad \lambda_F = -i\gamma_{\Omega,3} \lambda_F^T \gamma_{\Omega,3}^{-1}. \quad (5.12)$$

Then, the massless 33 fermions are also given by (5.11).

It remains to analyze the 93 and 39 states. From the mixed cylinder amplitude we deduce that in the NS sector there are neither tachyonic nor massless scalars. In the Ramond sector there is only one massless fermion of negative chirality in 93 and one of positive chirality in 39. The \mathbb{Z}_N projection on the 39 Chan-Paton factor is $\lambda = \gamma_{1,3} \lambda \gamma_{1,9}^{-1}$.

We also have to ensure that Ω^2 is realized properly. In general, for $p \neq 9$ states it must be that

$$\lambda = e^{i\pi(p-9)/4} \gamma_{\Omega,p} (\gamma_{\Omega,p}^T)^{-1} \lambda \gamma_{\Omega,9}^T \gamma_{\Omega,9}^{-1} . \tag{5.13}$$

For 39 states this gives a non-trivial constraint $\lambda = i\epsilon_3 \epsilon_9 \gamma_{N/2,3} \lambda$. The upshot is that in the \mathbb{Z}_4 orientifold with D3-branes we find mixed massless fermions

$$\left(\frac{1}{2}\right) [(\mathbf{16}, \mathbf{1}; \mathbf{16}, \mathbf{1}) + (\mathbf{1}, \mathbf{16}; \mathbf{1}, \mathbf{16})] . \tag{5.14}$$

The overall massless spectrum is anomaly-free as first observed in [7, 9].

Other configurations are viable. For example, $2k$ D3-branes plus their images can move to the bulk while $(64 - 4k)$ remain at the origin. The bulk branes give rise to an unitary group, in fact the full G_3 is $U(16 - k) \times U(16 - k) \times U(k)$ [9].

The T^2/\mathbb{Z}_4 orientifold is treated in the same fashion. Recall that fermions in $D=8$ are either $\mathbf{4}$ or $\bar{\mathbf{4}}$ of $SO(6)$. With all D7-branes at the origin, the 77, as well as the 99, massless fermions are a $\bar{\mathbf{4}}$ transforming under $U(16) \times U(16)$ as given in eq. (5.11). From strings between D9 and D7-branes there are tachyons. In the 97 sector there is also a massless fermion $\bar{\mathbf{4}}$ transforming as shown in eq. (5.14). The full spectrum has no $\text{tr } F^5$ anomalies. Different brane arrangements are feasible. For instance, with equal number of D7-branes on top of the four θ -fixed points, $G_7 = [U(4) \times U(4)]^4$, whereas with all branes in the bulk, $G_7 = U(16)$. Details of the spectrum can be worked out applying the rules explained above.

To finish we will briefly comment on states from closed strings. Since these orientifolds include the element $(-1)^{F_S}$ there are only bosons in the spectrum. The T^6/\mathbb{Z}_4 model has the interesting feature that closed tachyons are absent altogether because the Ω' projection eliminates the tachyon in the θ^2 sector and only massive scalars appear in other twisted sectors [7]. In the T^2/\mathbb{Z}_4 case the θ^2 tachyon is projected out too, but there are tachyons in other twisted sectors.

6. Final Comments

The principal motivation behind this work was to show that toroidal orientifolds of the non-supersymmetric type 0 strings can be treated on the same footing as type II orientifolds in which the orbifold point group explicitly breaks supersymmetry. This formulation has the advantage of allowing a transparent description of the resulting $D \leq 10$ theories using the same language employed in the study of supersymmetric orientifolds.

Modding by a non-supersymmetric \mathbb{Z}_N introduces subtleties that must be handled carefully. In some cases with N even, modular invariance demands that the order N be actually twice that of the geometrical rotation. In fact, in this situation the world-sheet fermions truly feel a \mathbb{Z}_N action. Hence, there are really N twisted sectors to be considered. Existence of an $\frac{N}{2}$ -th twisted sector in these cases explains typical features such as the appearance of a model-independent tachyon, the doubling of RR forms, and the cancellation of crosscap RR tadpoles when the Ω projection is used. Alternatively, with Ω' projection this tachyon is projected out and crosscap RR tadpoles remain. Furthermore, the non-supersymmetric actions require modified spin structures in the loop amplitudes that then

show up in tadpole cancellation conditions and in the GSO projections that determine the string states. We have striven to present clear prescriptions that can be easily applied in any example.

We have derived tadpole cancellation conditions that are also valid in the supersymmetric orientifolds and take into account the location of Dp -branes in the transverse DD directions. In models with D9 and D5-branes we obtained general expressions depending on the fixed set structure of the geometrical rotation. To our knowledge such detailed analysis had only been carried out before in specific examples [15, 19]. This is important because correct twisted tadpole formulas at all fixed sets are necessary to find solutions leading to anomaly-free spectra as it should. We have illustrated this issue with a \mathbb{Z}_8 orientifold in which tadpole conditions at the origin alone cannot imply anomaly cancellation.

Although we have not explored this possibility, the findings in this paper could be used to build semi-realistic examples. Upon T-duality, the models with D9 and D5-branes will include D3 and D7-branes as in the type of bottom-up constructions studied in refs. [25, 26]. Our results allow to find systematically the gauge theories living on D-branes at non-supersymmetric \mathbb{Z}_N singularities.

Another application is in the construction of $D=4$ orientifolds with flux [27]. In fact, we can already proffer a simple example. Consider, the \mathbb{Z}_4 orientifold with 64 D3-branes. In principle we can completely cancel O3-plane tadpoles by switching on fluxes of the RR and NSNS 3-forms with $N_{\text{flux}} = 64$. Thus, the D3-branes disappear and we are left just with D9-branes having $G_9 = U(16) \times U(16)$ and massless fermions given in (5.11). By itself this content is anomalous but the cubic anomaly can be precisely cancelled by the flux contribution [28, 29].

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A. Loop amplitudes and tadpoles

In this appendix we study the 1-loop amplitudes needed to compute tadpoles and the spectrum of states. We will proceed as in the supersymmetric case and will closely follow [19], limiting ourselves to describing the relevant changes particular to the non-supersymmetric setup.

Schematically, after going to the tree-level channel, the one-loop amplitudes give rise to divergences of the form

$$(T^{\text{NSNS}} - T^{\text{RR}}) \int_0^\infty d\ell, \tag{A.1}$$

where, say T^{RR} receives contributions $T_{\mathcal{K}}^{\text{RR}}$, T_{pq}^{RR} and T_p^{RR} from the amplitudes \mathcal{K} , \mathcal{C}_{pq} and \mathcal{M}_p . These divergences are due to the exchange of massless scalars in the tree-level closed string channels, NSNS or RR according to the superscripts. In absence of supersymmetry, the tadpoles T^{NSNS} and T^{RR} are not necessarily equal and only T^{RR} must cancel by consistency [10]. Below we discuss the Klein bottle, cylinder and Möbius strip amplitudes. Once the amplitudes are written explicitly it is straightforward to extract the RR tadpoles. The main results are collected and organized into tadpole cancellation conditions presented according to the existing Op -planes in sections 3, 4 and 5. These conditions are valid in the supersymmetric case as well.

A.1 Klein bottle amplitude

Being a closed string amplitude, \mathcal{K} includes a sum over twisted sectors and an orbifold projection. In general,

$$\mathcal{K} = \frac{V_D}{2N} \sum_{n,m=0}^N \int_0^\infty \frac{dt}{t} (4\pi^2 \alpha' t)^{-D/2} \mathcal{Z}_{\mathcal{K}}(\theta^n, \theta^m), \tag{A.2}$$

where

$$\mathcal{Z}_{\mathcal{K}}(\theta^n, \theta^m) = \text{Tr} \left(P_{\text{GSO}} \Omega_P \theta^m e^{-2\pi t[L_0 + \bar{L}_0]} \right). \tag{A.3}$$

The Virasoro operators are those of the θ^n -twisted sector. This trace does not contain the contribution of the non-compact momenta that already appears in (A.2). In particular, V_D is the regularized volume of the non-compact space-time. The GSO projection will be explained shortly. Since $\theta^n \rightarrow \theta^{N-n}$ under Ω_P , only the untwisted sector ($n = 0$) and the $\theta^{\frac{N}{2}}$ sector, when N is even, enter in the trace. The integral in (A.2) diverges as $t \rightarrow 0$, or equivalently as $\ell = 1/4t \rightarrow \infty$, as indicated in (A.1). Given the Ω_P insertion, these divergences are created by massless states in the sector twisted by θ^{2m} .

The bosonic contribution to $\mathcal{Z}_{\mathcal{K}}(\theta^n, \theta^m)$ contains a standard oscillator piece and for $n \neq 0$, a zero mode factor $\tilde{\chi}(\theta^n, \theta^m)$ equal to the number of points fixed simultaneously by θ^n and θ^m , in the sub-space where θ^n acts non-trivially. There is also a sum over quantized momenta and windings whenever θ^n and $\Omega\theta^m$ leave simultaneously invariant some sub-lattices of Λ^* and Λ . The invariant momentum sub-lattice is denoted Λ_I^* , its dimension I_P and its volume V_P , whereas the invariant winding sub-lattice is denoted Λ_I , its dimension I_W and its volume V_W (the dependence on n and m is understood and will be dropped for simplicity). For instance, in \mathbb{Z}_N^* examples, $\Lambda_I = \{0\}$, and moreover, either $\Lambda_I^* = \Lambda^*$ so that $V_D V_P$ is a 10-dimensional V_{10} , or $\Lambda_I^* = \{0\}$ and $V_P = 1$ by definition.

The fermionic contribution to (A.3) can be written in terms of ϑ functions but in the non-supersymmetric case we have to carefully take into account that in the θ^n twisted sector the spin-structure coefficients $s_{\alpha\beta}$ depend on n . In the untwisted sector the $s_{\alpha\beta}(0)$ are the usual ones (for type IIB):

$$s_{00}(0) = -s_{0\frac{1}{2}}(0) = -s_{\frac{1}{2}0}(0) = s_{\frac{1}{2}\frac{1}{2}}(0) = 1, \quad (\text{A.4})$$

for both left and right movers. This corresponds to the GSO projection

$$P_{\text{GSO}} = \frac{1}{4} \left(1 + (-1)^{f_L} \right) \left(1 + (-1)^{f_R} \right), \quad (\text{A.5})$$

where f_L, f_R , are world-sheet fermion numbers. We conclude that in the untwisted sector inserting Ω or $\Omega(-1)^{f_R}$ in the trace gives the same result because Ω alone forces $f_L = f_R$ in the GSO projection and then multiplying by $(-1)^{f_R}$ has no effect. In fact, for both Ω and Ω' we have

$$\mathcal{Z}_{\mathcal{K}}^{fer}(\mathbb{1}, \theta^m) = \sum_{\alpha, \beta=0, \frac{1}{2}} s_{\alpha\beta}(0) \prod_{a=0}^3 \frac{\tilde{\vartheta} \left[\begin{smallmatrix} \alpha \\ \beta + 2mv_a \end{smallmatrix} \right]}{\tilde{\eta}}. \quad (\text{A.6})$$

The tilde on $\tilde{\vartheta}$ and $\tilde{\eta}$ means that these functions have argument $e^{-4\pi t}$.

The $s_{\alpha\beta}(n)$ are the spin-structure coefficients that also appear in the torus amplitude. This is needed to have a consistent $\frac{1}{2}(1 + \Omega_P)$ projection. Now, it can be shown [20, 30] that modular invariance of the torus amplitude requires

$$s_{00}(n) = -s_{\frac{1}{2}0}(n) = 1 \quad ; \quad s_{0\frac{1}{2}}(n) = -s_{\frac{1}{2}\frac{1}{2}}(n) = -e^{-i\pi n S_v}. \quad (\text{A.7})$$

Incidentally, notice that $s_{0\frac{1}{2}}(0) = s_{0\frac{1}{2}}(N)$ implies (2.1).

Returning to $\mathcal{Z}_{\mathcal{K}}$, for N odd only the untwisted sector enters and with (A.6) we are finished. For N even we still need to analyze the $\theta^{\frac{N}{2}}$ sector. With $\Omega = \Omega_P$ the fermionic piece of the amplitude is just

$$\mathcal{Z}_{\mathcal{K}}^{fer}(\theta^{\frac{N}{2}}, \theta^m) = \sum_{\alpha, \beta=0, \frac{1}{2}} s_{\alpha\beta}(\frac{N}{2}) \prod_{a=0}^3 \frac{\tilde{\vartheta} \left[\begin{smallmatrix} \alpha + \frac{N}{2}v_a \\ \beta + 2mv_a \end{smallmatrix} \right]}{\tilde{\eta}}. \quad (\text{A.8})$$

Putting all pieces together we find that for $\Omega_P = \Omega$, the RR tadpole, conveniently normalized, is given by

$$T_{\mathcal{K}}^{\text{RR}}(n, m) = e^{-i\pi n S_v} 2^{D+I_P+I_W} \frac{V_D V_P}{V_W} \tilde{\chi}(\theta^n, \theta^m) \prod_{nv_j \in \mathbb{Z} + \frac{1}{2}} c(2mv_j + \frac{1}{2}) \prod_{\substack{nv_j \in \mathbb{Z} \\ 2mv_j \notin \mathbb{Z}}} 2 |\sin(2\pi mv_j)|, \quad (\text{A.9})$$

where we have defined

$$c(x) = \frac{|\cos \pi x|}{\cos \pi x}; \quad c(n + \frac{1}{2}) \equiv 0. \quad (\text{A.10})$$

When $nv_j \in \mathbb{Z} + \frac{1}{2}$ and $2mv_j \in \mathbb{Z}$, this Klein tadpole does vanish. For instance, $T_{\mathcal{K}}^{\text{RR}}(\frac{N}{2}, \frac{N}{2}) = 0$ in the \mathbb{Z}_{even} examples with O5-planes.

In some cases, in particular in the $\mathbb{Z}_{\text{even}}^*$ and the \mathbb{Z}_4 in $D = 8, 4$ in (2.3), $\frac{N}{2}v$ happens to be an $\text{SO}(8)$ weight in the vector class and indeed, $\theta^{\frac{N}{2}}$ is equivalent to $(-1)^{F_S}$. Then, closed states in the $\theta^{\frac{N}{2}}$ sector are rather characterized by a weight $r' = r + \frac{N}{2}v$. Moreover, in these examples, $s_{0\frac{1}{2}}(\frac{N}{2}) = s_{\frac{1}{2}\frac{1}{2}}(\frac{N}{2}) = 1$. Hence, the sum over spin structure demands $\sum_a r'_a = \text{even}$ and the GSO projection effectively changes to

$$P_{\text{GSO}}^- = \frac{1}{4} \left(1 - (-1)^{f_L}\right) \left(1 - (-1)^{f_R}\right). \quad (\text{A.11})$$

Therefore, the difference between inserting Ω and $\Omega(-1)^{f_R}$ is an overall minus sign, i.e.

$$\mathcal{Z}_{\mathcal{K}}^{\Omega'}(\theta^{\frac{N}{2}}, \theta^m) = -\mathcal{Z}_{\mathcal{K}}^{\Omega}(\theta^{\frac{N}{2}}, \theta^m). \quad (\text{A.12})$$

We conclude that when $\Omega_P = \Omega'$, the tadpole $T_{\mathcal{K}}^{\text{RR}}(\frac{N}{2}, m)$ picks an extra minus sign. Clearly, this minus sign affects the type of divergences. We will see that in these examples with Ω projection the RR tadpoles due to $\mathcal{Z}_{\mathcal{K}}(\mathbb{1}, \theta^m)$ and $\mathcal{Z}_{\mathcal{K}}(\theta^{\frac{N}{2}}, \theta^m)$ cancel each other but there are left-over NSNS tadpoles. With Ω' projection the opposite happens. Furthermore, the overall minus sign in $\mathcal{Z}_{\mathcal{K}}^{\Omega'}(\theta^{\frac{N}{2}}, \theta^m)$ helps to get rid of tachyons. When $\frac{N}{2}v$ is an $\text{SO}(8)$ weight in the vector class, it is easy to show that the mass formula and the GSO projection always allow a tachyon in the $\theta^{\frac{N}{2}}$ sector, namely the state with $r' = 0$. If $\Omega_P = \Omega$, the tachyon survives the orientifold projection but it is projected out for $\Omega_P = \Omega'$. The reason is precisely the relative minus sign between the Klein bottle trace and the corresponding torus trace [3, 11].

When $\frac{N}{2}v$ is such that $\theta^{\frac{N}{2}}$ is not equivalent to $(-1)^{F_S}$ we cannot conclude that there is a change of GSO projection in this sector. Thus, the Klein amplitude is the same for both Ω and Ω' as in supersymmetric orientifolds.

A.2 Cylinder amplitudes

Typically, the Klein bottle amplitude has divergences proportional to V_{10} or V_D that can be cancelled by adding D9-branes. For the \mathbb{Z}_N^* these are the only type of divergences. For other actions lower dimensional branes are also needed. In particular, all \mathbb{Z}_6 and the $D=6$ \mathbb{Z}_4 in (2.3), have D5₁-branes. For the \mathbb{Z}_4 in $D=8$ and $D=4$ there are respectively D7₃ and D3-branes.

Open strings between branes lead to cylinder amplitudes given by

$$\mathcal{C}_{pq} = \frac{V_D}{2N} \sum_{m=0}^N \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-D/2} \mathcal{Z}_{pq}(\theta^m), \tag{A.13}$$

where

$$\mathcal{Z}_{pq}(\theta^m) = \text{Tr} (P_{\text{GSO}} \theta^m e^{-2\pi t L_0}). \tag{A.14}$$

The trace is over open string states with Dp and Dq branes at the endpoints. The integral has divergences of type (A.1) due to massless states in the sector twisted by θ^m .

Consider first the \mathcal{Z}_{pp} cylinders. In this case the boundary conditions are either Neumann-Neumann (NN) for longitudinal or Dirichlet-Dirichlet (DD) for transverse directions. The fermionic contribution is then just

$$\mathcal{Z}_{pp}^{fer}(\theta^m) = \sum_{\alpha, \beta=0, \frac{1}{2}} s_{\alpha\beta}(0) \prod_{a=0}^3 \frac{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta + m v_a \end{smallmatrix} \right]}{\eta}, \tag{A.15}$$

with $s_{\alpha\beta}(0)$ given in (A.4). The way the Chan-Paton degrees of freedom enter in the trace depends on the boundary conditions. For instance, for $p = 9$ all directions are NN and there is factor $(\text{Tr} \gamma_{m,9})^2$. To study other Dp -branes we denote by θ_{NN}^m and θ_{DD}^m the restriction of θ^m to the NN and DD directions respectively. The Dp -branes are located at some points in the transverse directions with DD boundary conditions. If these points are not fixed under θ_{DD}^m , the trace vanishes. To obtain a non-zero trace there must be Dp -branes sitting at fixed points of θ_{DD}^m . The number of such points will be denoted $\tilde{\chi}(\theta_{\text{DD}}^m)$. Then, $\mathcal{Z}_{pp}(\theta^m)$ will have a factor

$$\sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} (\text{Tr} \gamma_{m,p,J})^2. \tag{A.16}$$

Recall that in general $\tilde{\chi}(\theta^m)$ is given by

$$\tilde{\chi}(\theta^m) = \prod_{m v_j \notin \mathbb{Z}} 4 \sin^2 \pi m v_j. \tag{A.17}$$

The boundary conditions also determine the type of lattice sums that can appear in the bosonic part of $\mathcal{Z}_{pp}(\theta^m)$. In the NN directions there will be a sum over quantized momenta if Λ^* has a sublattice, of volume V_P^{NN} , invariant under θ_{NN}^m . In the DD directions there can instead be windings. The volume of the Λ sub-lattice invariant under θ_{DD}^m will be denoted V_W^{DD} .

From the \mathcal{C}_{pp} amplitude we find the RR tadpole:

$$T_{pp}^{\text{RR}}(m) = \frac{V_D V_P^{\text{NN}}}{V_W^{\text{DD}}} \prod_{m v_j \notin \mathbb{Z}} 2 |\sin \pi m v_j| \sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} (\text{Tr} \gamma_{m,p,J})^2. \tag{A.18}$$

To compute the RR tadpoles due to \overline{Dp} -branes we just need to take into account the change of sign in the RR charge. Concretely, the RR tadpoles from $\mathcal{C}_{\overline{p}\overline{p}}$, and $\mathcal{C}_{p\overline{p}}$ plus $\mathcal{C}_{\overline{p}p}$,

are obtained from (A.18) replacing $(\text{Tr } \gamma_{m,p,J})^2$ by $(\text{Tr } \gamma_{m,\bar{p},J})^2$ and $-2\text{Tr } \gamma_{m,p,J}\text{Tr } \gamma_{m,\bar{p},J}$ respectively. Thus, the net effect of $\overline{\text{D}p}$ -branes amounts to substituting

$$\text{Tr } \gamma_{m,p,J} \rightarrow \text{Tr } \gamma_{m,p,J} - \text{Tr } \gamma_{m,\bar{p},J} \quad (\text{A.19})$$

in (A.18), as in orientifolds with supersymmetric \mathbb{Z}_N action [12].

For our purposes, we do not need to analyze the generic \mathcal{C}_{pq} , only \mathcal{C}_{p9} in which NN or DN boundary conditions arise. Coordinates Y^i with mixed DN boundary conditions have oscillator expansions with half-integer modes. For the fermions Ψ^i world-sheet supersymmetry then implies integer modes in NS and half-integer in R. In practice, the upper characteristic of the ϑ functions in (A.15) must be replaced by $\alpha + \frac{1}{2}$ for all DN directions. Effectively, \mathcal{Z}_{p9} conforms to the amplitude of a sector twisted by v_{p9} with entries $\frac{1}{2}$ in the DN directions. Thus, necessarily the spin structure coefficients must be that of a twisted sector. When $p = 5$, we can take $v_{59} = \frac{N}{2}v$, so that $s_{0\frac{1}{2}} = -e^{-i\pi N S v/2}$. On the other hand, for $p = 7, 3$, $v_{p9} = \frac{N}{4}v$ and $s_{0\frac{1}{2}} = -e^{-i\pi N S v/4}$. These modified spin structure coefficients will show up in the RR tadpoles because the $\alpha = 0$, $\beta = \frac{1}{2}$ term in the loop channel goes into RR in the tree-level channel. Another relevant effect is the change of the GSO projection for $p9$ states.

We now consider $p = 5$ for concreteness. If Y^j is DN and $mv_j \in \mathbb{Z}$, the RR tadpole arising from $\mathcal{Z}_{59}(\theta^m)$ vanishes, the typical case being $m = 0$. Otherwise we find:

$$T_{95+59}^{\text{RR}}(m) = 2 e^{-i\pi \frac{N}{2} S v} V_D V_P^{\text{NN}} \prod_{\substack{\text{NN} \\ mv_j \notin \mathbb{Z}}} 2 |\sin(\pi m v_j)| \prod_{\text{DD}} s(m v_j) \sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} \text{Tr } \gamma_{m,9} \text{Tr } \gamma_{m,5,J}, \quad (\text{A.20})$$

where we have defined

$$s(x) = \frac{|\sin \pi x|}{\sin \pi x} \quad ; \quad s(n) \equiv 0. \quad (\text{A.21})$$

Notice that the DN (ND) directions for 59 (95) strings are the same as the DD directions for the corresponding 55 strings. All cylinder tadpoles have the same volume dependence if $mv_j \notin \mathbb{Z}$ for the DD coordinates, which in particular occurs when m is odd and also when $m = \frac{N}{2}$. We can then sum the tadpoles to obtain

$$\sum_{p,q=9,5} T_{pq}^{\text{RR}}(m) = V_D V_P^{\text{NN}} \sqrt{\frac{\tilde{\chi}(\theta_{\text{NN}}^m)}{\tilde{\chi}(\theta_{\text{DD}}^m)}} \sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} \left| \text{Tr } \gamma_{m,9} + \xi(mv) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^m)} \text{Tr } \gamma_{m,5,J} \right|^2. \quad (\text{A.22})$$

Here we have defined

$$\xi(mv) = e^{-i\pi \frac{N}{2} S v} \prod_{\text{DD}} s(mv_j). \quad (\text{A.23})$$

Also, by convention $\tilde{\chi}(\theta_{\text{NN}}^m) \equiv 1$ when $mv_j \in \mathbb{Z}$ for the NN directions.

A.3 Möbius strip amplitudes

The Möbius strip amplitudes are given by:

$$\mathcal{M}_p = \frac{V_D}{2N} \sum_{m=0}^N \int_0^\infty \frac{dt}{t} (8\pi^2 \alpha' t)^{-D/2} \mathcal{Z}_p(\theta^m), \quad (\text{A.24})$$

where

$$\mathcal{Z}_p(\theta^m) = \text{Tr} \left(P_{\text{GSO}} \Omega_P \theta^m e^{-2\pi t L_0} \right). \quad (\text{A.25})$$

The trace is over open string states with Dp branes at both endpoints. Due to the Ω_P insertion the integral has divergences of type (A.1) due to massless states in the sector twisted by θ^{2m} . The resulting RR tadpoles will be denoted $T_p^{\text{RR}}(\theta^m)$.

To evaluate $\mathcal{Z}_p(\theta^m)$ we start from $\mathcal{Z}_{pp}(\theta^m)$ and implement the action of Ω_P inserted in the trace. Notice that being an open string amplitude, Ω and Ω' give the same result. For directions with NN boundary conditions Ω acts on bosonic and fermionic oscillators, of frequency r , as $\alpha_r \rightarrow e^{i\pi r} \alpha_r$ and $\psi_r \rightarrow e^{i\pi r} \psi_r$. In terms of the expansion variable $q = e^{-2\pi t}$ this amounts to

$$q \rightarrow Q = e^{-i\pi} q. \quad (\text{A.26})$$

Furthermore, Ω multiplies the NS vacuum by $e^{-i\pi/2}$ and the R vacuum by -1 . Then, leaving out possible sums over quantized momenta when $mv_j \in \mathbb{Z}$, the Möbius amplitude for D9-branes takes the form

$$\mathcal{Z}_9(\theta^m) = - \left(\text{Tr} \gamma_{\Omega m,9}^{-1} \gamma_{\Omega m,9}^T \right) \sum_{\alpha,\beta=0,\frac{1}{2}} s_{\alpha\beta}(0) \prod_{a=0}^3 \frac{-2 \sin \pi m v_a \eta_Q}{\vartheta_Q \left[\frac{\frac{1}{2}}{\frac{1}{2} + m v_a} \right]} \frac{\vartheta_Q \left[\frac{\alpha}{\beta + m v_a} \right]}{\eta_Q}, \quad (\text{A.27})$$

where the subscript Q means that the ϑ and η functions have Q as variable. We can extract the divergence by performing modular transformations [11]. The resulting tadpole is given in (3.5).

For directions with DD boundary conditions there is an extra minus sign when Ω acts on oscillators, i.e. $\alpha_r \rightarrow -e^{i\pi r} \alpha_r$ and $\psi_r \rightarrow -e^{i\pi r} \psi_r$. This has the effect of shifting by $\frac{1}{2}$ the lower characteristic of the ϑ functions involved in the trace. To be more specific, consider the case of D5-branes. Then, as mentioned previously, the vector $\frac{N}{2}v$ has precisely half-integer entries in DD but integer in the NN directions. We find that the consistent way of shifting the lower characteristic is by adding $\frac{N}{2}v$. The trace will then vanish identically in the supersymmetric cases as it should. Besides, changing the sign of v gives the same result as expected. In the end we arrive at the trace

$$\begin{aligned} \mathcal{Z}_5(\theta^m) = & - \sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} \left(\text{Tr} \gamma_{\Omega m,5,J}^{-1} \gamma_{\Omega m,5,J}^T \right) \\ & \times \sum_{\alpha,\beta=0,\frac{1}{2}} s_{\alpha\beta}(0) \prod_{a=0}^3 \frac{-2 \sin \pi \left(m + \frac{N}{2} \right) v_a \eta_Q}{\vartheta_Q \left[\frac{\frac{1}{2}}{\frac{1}{2} + m v_a + \frac{N}{2} v_a} \right]} \frac{\vartheta_Q \left[\frac{\alpha}{\beta + m v_a + \frac{N}{2} v_a} \right]}{\eta_Q}. \end{aligned} \quad (\text{A.28})$$

Here we have not written down possible sums over quantized momenta when $mv_j \in \mathbb{Z}$ for NN directions or over quantized windings when $mv_j \in \mathbb{Z} + \frac{1}{2}$ for DD directions. The

resulting tadpole turns out to be

$$\begin{aligned}
 T_5^{\text{RR}}(m) &= -2^{\frac{1}{2}(D+I_P+I_W+2)} \frac{V_D V_P^{\text{NN}}}{V_W^{\text{DD}}} \epsilon_5 \sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^m)} \text{Tr } \gamma_{2m,5,J} \\
 &\times \prod_{a=0}^3 c(mv_a + \frac{N}{2}v_a) \prod_{(m+\frac{N}{2})v_j \notin \mathbb{Z}} 2 |\sin(\pi(m + \frac{N}{2})v_j)| \quad (\text{A.29})
 \end{aligned}$$

where we have used (2.8).

We finally work out some explicit results essential to compute the tadpole cancellation condition when the \mathbb{Z}_N action takes the form (4.1) and there are O9 and O5-planes. Concretely, since we need the Möbius tadpoles due to massless states in a $\theta^{2\ell}$ sector, we will consider $m = \ell$ and $m = \ell + \frac{N}{2}$ in the above expressions. When $lv_j \notin \mathbb{Z} + \frac{1}{2}$, $\forall j$, all Möbius tadpoles $T_p^{\text{RR}}(\ell + k)$, $k = 0, \frac{N}{2}$, have the same volume dependence $V_D V_P^{\text{NN}}$. Moreover, they can be conveniently added according to the sector. Concretely,

$$\begin{aligned}
 \sum_{p=9,5} T_p^{\text{RR}}(\ell) &= -2^{\frac{1}{2}(D+I_P+2)} V_D V_P^{\text{NN}} \sqrt{\frac{\tilde{\chi}(\theta_{\text{NN}}^\ell) \tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})}{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})}} \epsilon_9 \prod_{a=0}^3 c(\ell v_a) \times \\
 &\sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^\ell)} \left[\text{Tr } \gamma_{2\ell,9} - \epsilon_9 \epsilon_5 \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})} \text{Tr } \gamma_{2\ell,5,J} \right]; \quad (\text{A.30}) \\
 \sum_{p=9,5} T_p^{\text{RR}}(\ell + \frac{N}{2}) &= -2^{\frac{1}{2}(D+I_P+2)} V_D V_P^{\text{NN}} \sqrt{\frac{\tilde{\chi}(\theta_{\text{NN}}^\ell) \tilde{\chi}(\theta_{\text{DD}}^\ell)}{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})}} \epsilon_9 \delta_9 \prod_{a=0}^3 c(\ell v_a + \frac{N}{2}v_a) \times \\
 &\sum_{J=1}^{\tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})} \left[\text{Tr } \gamma_{2\ell,9} + \delta_9 \delta_5 \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})} \text{Tr } \gamma_{2\ell,5,J} \right]. \quad (\text{A.31})
 \end{aligned}$$

In these expressions we have substituted the relation

$$\prod_{a=0}^3 c(\ell v_a + \frac{N}{2}v_a) c(\ell v_a) = -e^{-i\pi \frac{N}{2} S_v} \prod_{\text{DD}} s(2\ell v_j) = -\xi(2\ell v). \quad (\text{A.32})$$

We have also used the trigonometric identity

$$\tilde{\chi}(\theta_{\text{DD}}^{2\ell}) = \tilde{\chi}(\theta_{\text{DD}}^\ell) \tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}}) \quad (\text{A.33})$$

that follows from the explicit form of θ and (A.17). We further remark that for all crystallographic actions of type (4.1) and for ℓ such that $lv_1 \notin \mathbb{Z} + \frac{1}{2}$, there is an equality $\tilde{\chi}(\theta_{\text{NN}}^{2\ell}) = \tilde{\chi}(\theta_{\text{NN}}^\ell)$.

Clearly, the Möbius tadpoles (A.30) and (A.31) are equally distributed over the fixed points of θ_{DD}^ℓ and $\theta_{\text{DD}}^{\ell+\frac{N}{2}}$ respectively. The origin, labelled by $J = 1$, is a simultaneous fixed

point so that its share of Möbius tadpoles, denoted $T_{\mathcal{M},1}^{\text{RR}}(\ell)$, is a sum of contributions from both sectors. We find

$$\begin{aligned}
 T_{\mathcal{M},1}^{\text{RR}}(\ell) = & 2^{\frac{1}{2}(D+I_P+2)} V_D V_P^{\text{NN}} \sqrt{\frac{\tilde{\chi}(\theta_{\text{NN}}^\ell)}{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})}} \epsilon_9 \prod_{a=0}^3 c(\ell v_a) \left[\sqrt{\tilde{\chi}(\theta_{\text{DD}}^{\ell+\frac{N}{2}})} - \delta_9 \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^\ell)} \right] \\
 & \times \left[\text{Tr } \gamma_{2\ell,9} + \xi(2\ell v) \sqrt{\tilde{\chi}(\theta_{\text{DD}}^{2\ell})} \text{Tr } \gamma_{2\ell,5,1} \right]. \tag{A.34}
 \end{aligned}$$

Other common fixed points receive the same share.

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